**Summer school on $C^*$-algebras and their interplay with dynamical systems**
**31 May - 4 June, 2010, Nordfjordeid, Norway**

This summer school was organised with the support of the Research Council of Norway, through the project "Operator Algebras", project 191195/V30, and of NordForsk through the research network "Operator Algebra and Dynamics".

The organisers of the summer school were Toke M. Carlsen (NTNU), Magnus B. Landstad (NTNU) and Nadia S. Larsen (University of Oslo). The scientific committee consisted, in addition to Carlsen, Landstad and Larsen, also of Erik Bedos and Sergey Neshveyev (University of Oslo).

The venue of the summer school was the "Sophus Lie Conference Center" located in Nordfjordeid, on the west coast of Norway. Sophus Lie (1842-1899) was born in Nordfjordeid and stands today as one of the most prominent figures in the gallery of Norwegian mathematicians.

The summer school was directed at Ph.D.-students and postdocs, and consisted of three mini-courses and 12 contributed talks by participants.

We experienced great interest for attendance in the summer school. In all 34 mathematicians attended the school (21 Ph.D.-students, 3 postdocs and 10 tenured staff). Geographically, the distribution of participants was impressive: 5 (Denmark), 1 (the Faroe Islands), 3 (Sweden), 10 (Norway), 3 (Germany), 1 (Italy), 1 (UK), 7 (USA), 2 (Japan) and 1 (Australia).

## Scientific Programme

### Mini-courses:

**(1)** Steve Kaliszewski and John Quigg (Arizona State University, USA) lectured on *Categorical perspectives in noncommutative duality*,

**(2)** Aidan Sims (University of Wollongong, Australia) lectured on *Higher-rank graphs and their $C^*$-algebras*, and

**(3)** Andreas Thom (Universität Leipzig, Germany) lectured on *Stability of unitary representations.*

**(1)** In their first lecture, Steve Kaliszewski and John Quigg introduced the audience to classical Landstad duality and paved the way to reformulating it as categorical Landstad duality for reduced crossed products from group actions on $C^*$-algebras. The reformulation has the advantage that it carries along more information on the answer as the categorical perspective requires certain techniques of proof which, when employed, provide new insight into the original question. A bit more explicitly, Landstad's original and highly nontrivial answer to the question of when a given $C^*$-algebra $B$ is isomorphic to the reduced crossed product of a system $(A, G, \alpha)$ can be phrased as an answer to the context of objects in the category of actions of $G$ on $C^*$-algebras (with appropriate homomorphisms as morphisms). The new perspective lifts this answer to also cover the morphisms in the category, and yields a more robust theory. In proper language, CLDA says that the crossed product functor is an equivalence from the category of actions of $G$ to the comma category of normal coactions of $G$ under the canonical coaction $\delta_G$ on $C^*(G)$.

In the second lecture Steve started by highlighting the crucial ingredients of the proof of the categorical Landstad duality for actions, and explained the dual result for coactions. John then made the passage to the categorical Landstad duality for full crossed products from group actions on $C^*$-algebras, and introduced the functor Nor which in a certain sense takes the functor CP (crossed product) to the functor RCP (reduced crossed product). John also indicated how the new categorical perspective was succesfully employed to shed light on the "Enchilada" category of $C^*$-algebras with isomorphism classes of $C^*$-correspondences as morphisms.

In the third lecture, Steve started on the proof of the fact that the comma categories of maximal and normal coactions are equivalent. To achieve this, one looks into the category of coactions of $G$ with equivariant morphisms, and there identifies two subcategories, one of normal and one of maximal coactions, and functors Nor and Max which form an adjoint pair. In the final lecture John outlined the proof of the aforementioned equivalence of subcategories and convinced us of the validity of categorical Landstad duality for full crossed products. John ended the lecture by posing the following question: is there another naturally arising category which contains two subcategories with properties of the sort enjoyed by the maximal and normal coactions in the category of all coactions?

**(2)** Aidan Sims started by defining higher rank graphs and their genesis from coloured skeletons based on directed graphs. Given a non-negative integer $k$, a $k$-graph $\Lambda$ is a countable category equipped with a degree functor $d$ into $\mathbb{N}^k$ such that every $\lambda$ in $\Lambda$ with degree $m+n$ factorises uniquely as $\mu\nu$ with the degrees of $\mu$ and $\nu$ equal to $m$ and $n$, respectively. Aidan presented the construction of $k$-graphs from a complete and associative collection $\mathcal{C}$ of squares for a $k$-coloured graph by using $\mathcal{C}$-compatible coloured-graph morphisms. To a $k$-graph which satisfies technical hypotheses (locally convex and row-finite), a Cuntz-Krieger $\Lambda$-family in a $C^*$-algebra $B$ is an assignment of projections in $B$ to the vertices, i.e. the objects in $\Lambda$, and partial isometries $t_\lambda$, $\lambda \in \Lambda$, which satisfy a number of relations, and one proves that there exists a universal $C^*$-algebra $C^*(\Lambda)$ generated by a nonzero Cuntz-Krieger $\Lambda$-family. This $C^*$-algebra comes equipped with a canonical action of $\mathbb{T}^k$, the gauge-action. Aidan sketched the proof of the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ and highlighted the first appearance of a result in this spirit in work of an Huef and Raeburn on Cuntz-Krieger algebras. Then he focused on the other type of uniqueness result, namely the Cuntz-Krieger uniqueness theorem, which gives criteria for injectivity of representations of $C^*(\Lambda)$ only in terms of the $k$-graph. Specifically, the representation arising from a Cuntz-Krieger $\Lambda$-family with nonzero projections on vertices is injective when the $k$-graph is aperiodic. When the $k$-graph in addition is cofinal, the $C^*$-algebra $C^*(\Lambda)$ is simple.

In his second lecture Aidan defined $k$-morphs between $k$-graphs, and illustrated their role in linking the two $k$-graphs into a $(k+1)$-graph. He gave illuminating examples of such linking graphs from cycles. When the $k$-morph links two copies of the same $k$-graph $\Lambda$ the linking graph is a kind of crossed product of $\Lambda$ with $\mathbb{N}$ by the $k$-morph, and its $C^*$-algebra can be recovered up to isomorphism as the Cuntz-Pimsner algebra of a certain $C^*(\Lambda)$-$C^*(\Lambda)$-correspondence. Then Aidan introduced rank-2 Bratteli diagrams; these are 2-graphs arising from a sequence of 1-graphs, each of which is a finite union of cycles, and compatible 1-morphs. It turns out that for a rank-2 Bratteli diagram $\Gamma$, the resulting $C^*$-algebra $C^*(\Gamma)$ is an A$\mathbb{T}$-algebra. Aidan then presented a result which gives necessary conditions for $C^*(\Gamma)$ to be simple and with real rank zero, thus placing $C^*(\Gamma)$ in an attractive subclass of classifiable $C^*$-algebras, in Elliott's sense.

In the final lecture Aidan discussed connections between coaction crossed products and $k$-graphs. As applications he showed that the crossed product of $C^*(\Lambda)$ by the gauge-action is an AF algebra, and recovered the Bunce-Deddens algebra with supernatural number $2^\infty$ as a coaction crossed product of $C(\mathbb{T})$ by the 2-adic numbers. The moral of the story is that $k$-graphs and constructions based on them are versatile tools for studying $C^*$-algebras.

**(3)** Andreas Thom began by explaining why we need to study infinite dimensional unitary representations of groups. He presented the by now classical theorem which states that a locally compact abelian group can be recovered as those natural transformations between the functors $\mathrm{Rep}_G$ and $\mathrm{Rep}_{\mathbb{Z}}$ that are compatible with direct sums, tensor products and conjugation (we recall that for a Hilbert space $H$, $\mathrm{Rep}_G(\mathrm{H})$ is the topological space consisting of homomorphisms from $G$ to the unitary group of $H$ endowed with the compact-open topology and $\mathrm{Rep}_{\mathbb{Z}}(H)$ is $U(H)$ with its strong topology). For compact groups, the Tannaka-Krein theory gives the same result involving only finite-dimensional representations. In the 60's, Chu showed that a virtually abelian group (i.e. a group containing a finite-index abelian subgroup) can be recovered from natural transformations between $\mathrm{Rep}_G$ and $\mathrm{Rep}_{\mathbb{Z}}$ only requiring finite-dimensional representations. A question left open was whether one could go beyond virtually abelian groups. More specifically, would Chu's result hold for the free group on two generators? Andreas proved that the class of virtually abelian groups is precisely the class for which one can have Chu's theorem. The main ingredient of proof is a technical result describing convergence to identity, uniformly on finite dimensional representations, of a sequence of non-trivial elements in the free group on two generators.

In his second lecture, Andreas introduced us to the stability of unitary representations. He defined amenable groups and proved the equivalence of amenability (in the classical sense of admitting a finite weight, finitely additive invariant measure) with the lack of paradoxical decompositions and Følner's condition. He then proved the following theorem of Kazhdan: for an amenable group, every almost unitary representation is close to a unitary representation. The proof presented uses an algorithm of Shtern for producing unitary representations.

In the third lecture, Andreas started by proving that for an amenable group there is, for any positive $\varepsilon$, a positive $\delta$ such that unitary representations that are close within $\delta$ are conjugate by a unitary which is $\varepsilon$-close to the identity on the Hilbert space. This result places amenable groups in the class of deformation rigid groups. The class of amenable groups is both nice and large. Outside it there are the groups containing a free subgroup, and more recently progress has been made on the Burnside groups; these are more elusive non-amenable groups with no free subgroups. A theorem of

Dixmier asserts that for an amenable group, every uniformly bounded representation is unitarisable (i.e. there is an invertible operator on the Hilbert space which conjugates the representation into a representation by unitaries). Dixmier's conjecture asserts that amenable groups are the only ones with this behaviour. Andreas sketched the proof of a result of Monod and Ozawa showing that (many) Burnside groups are not unitarisable. The final theorem showed that a unitarisable group must be deformation rigid. Dixmier's conjecture is still open, but perhaps nearer elucidation!

**Social Programme**

All participants stayed at the "Sophus Lie Conference Center" which is located within the premises of Fiordane Folkehøskole in the middle of Nordfjordeid. This created a good atmosphere where in very short time all participants met each other and could easily talk to each other during breaks and in the evenings.

In the afternoon of June 1 we took a bus trip to the famous Briksdal glacier. The weather was excellent with a clear sky offering a fantastic view of the rugged Norwegian coast alongside the innermost arms of the Nordfjord. At Briksdal we walked to see the glacier close by. This time of the year being just the very start of the high-season, we found our way on the trail among tourists from only one cruise ship.

We all agreed that the summer school had been a successful event, and that we would like to have such a meeting again, at the same location, if not yearly then as often as may be.

N. Larsen, June 22, 2010