An explanation for generalized failure criteria

by
Antony Mossop
Shell / NAM
Netherlands

- Stress and Deformation

- Failure and Yield Surfaces

- Laboratory Measurements

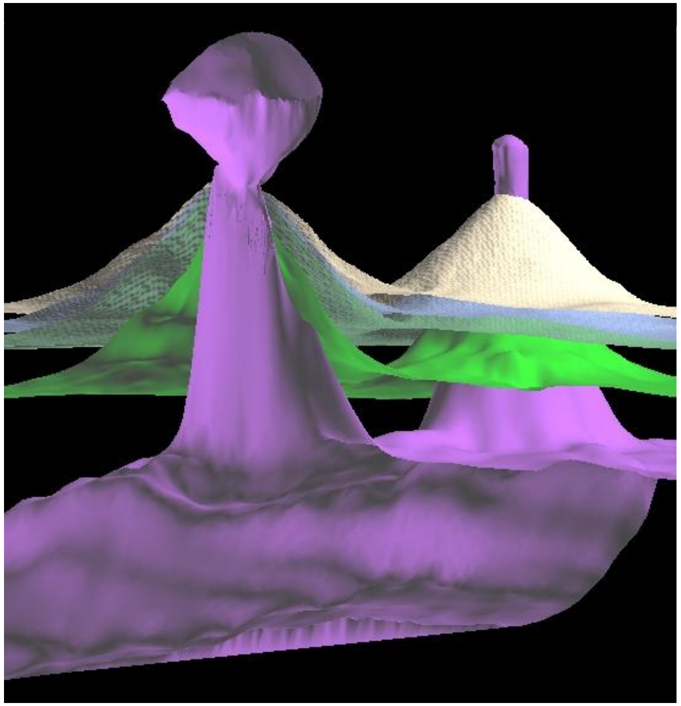
- Multiaxial Conditions

- The 'Correct' Shear Stress Measure

- Other Applications

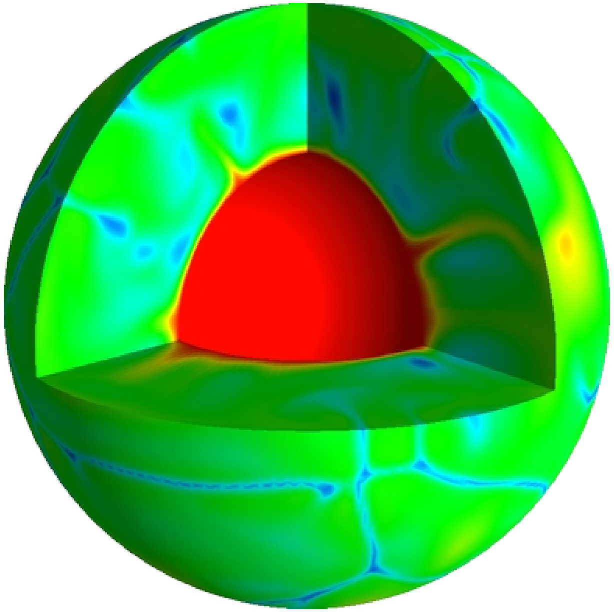
- Conclusions

Understanding how rocks and other geo-materials deform under stress is central to the discipline of geomechanics.



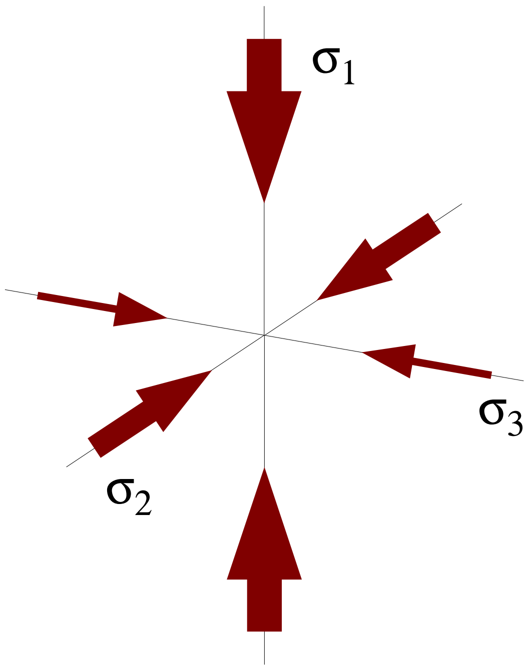
Salt Flow

Failure Processes

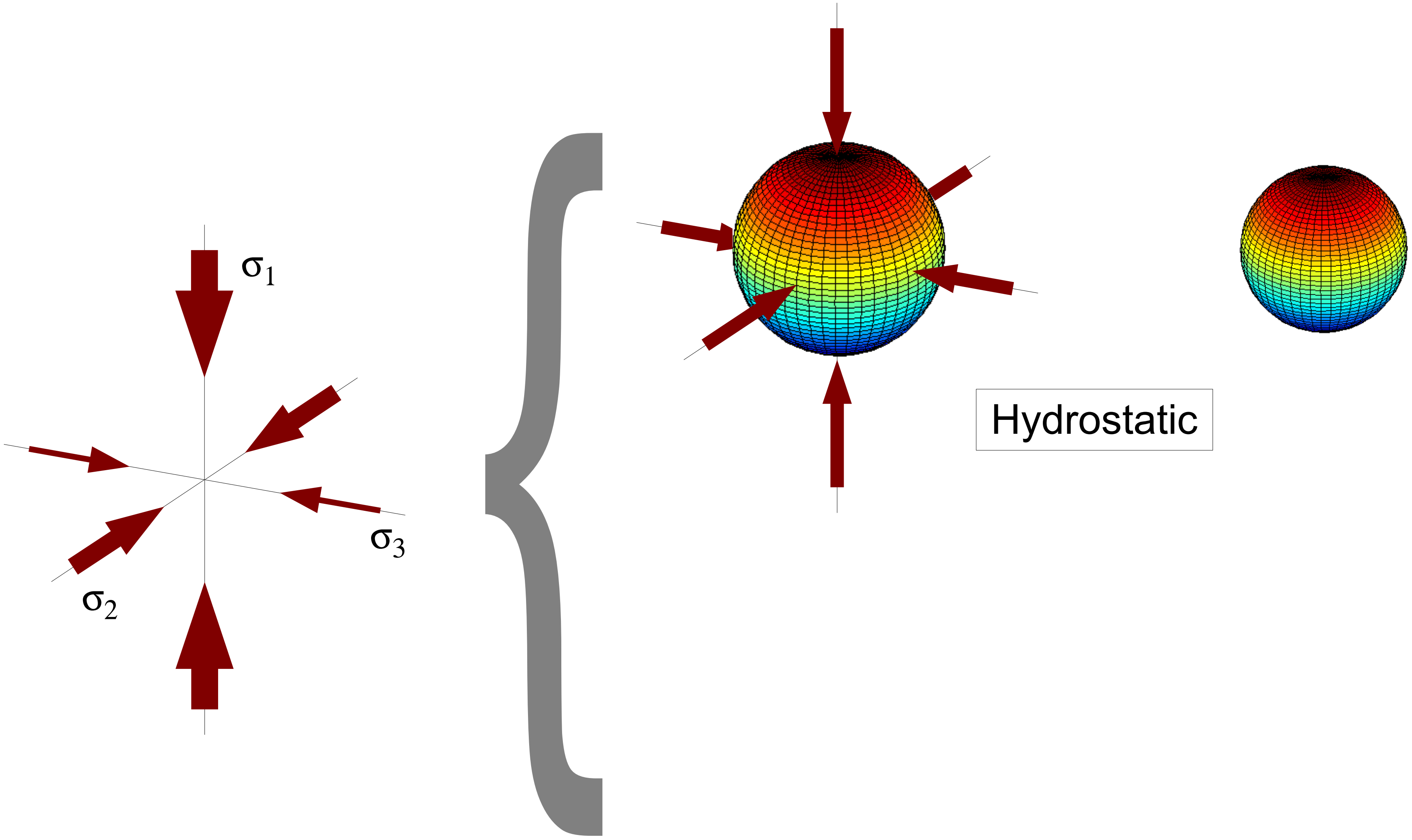


Mantle Modeling

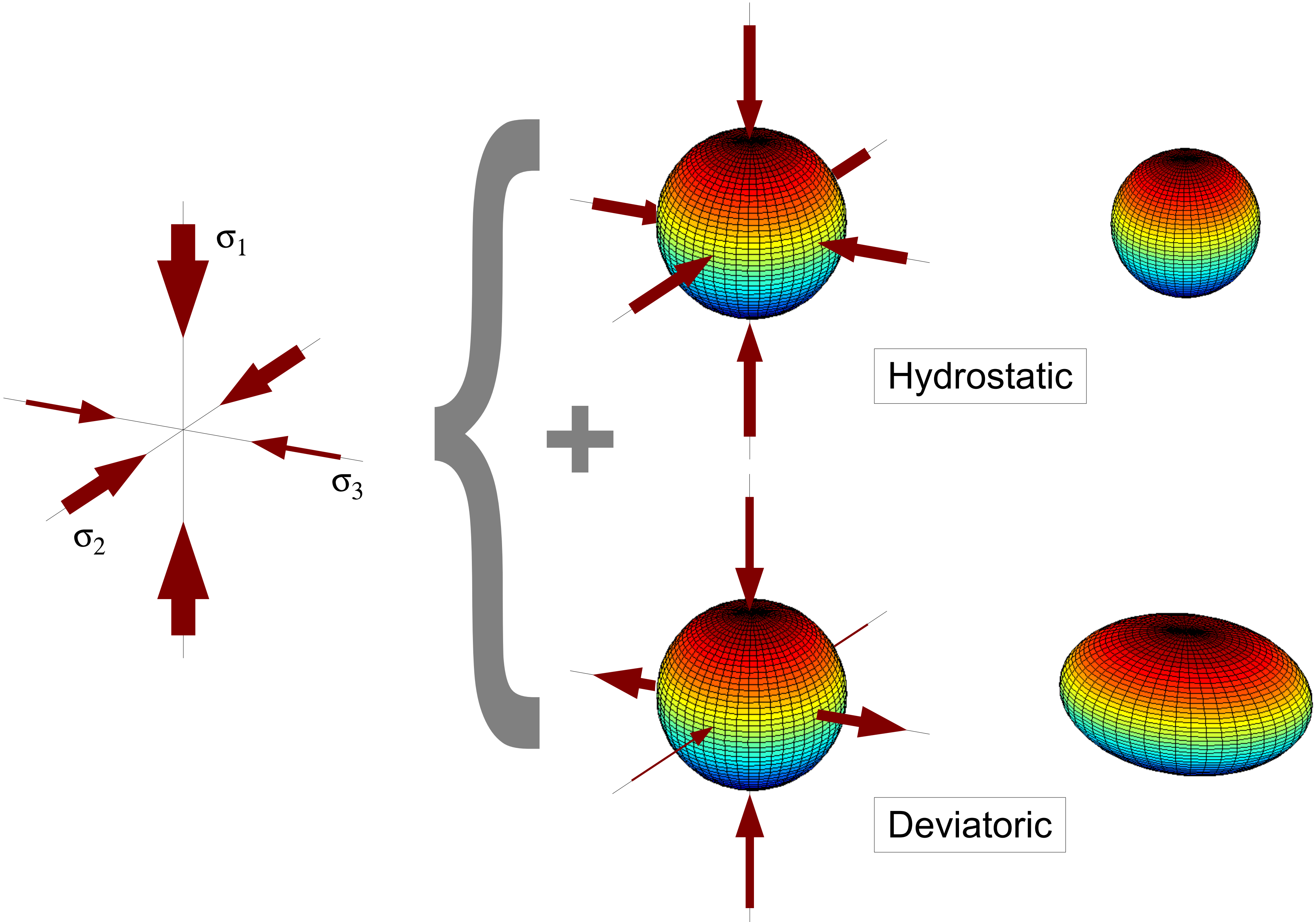
Stress though, is not a single number (scalar), it's six numbers (symmetric tensor).



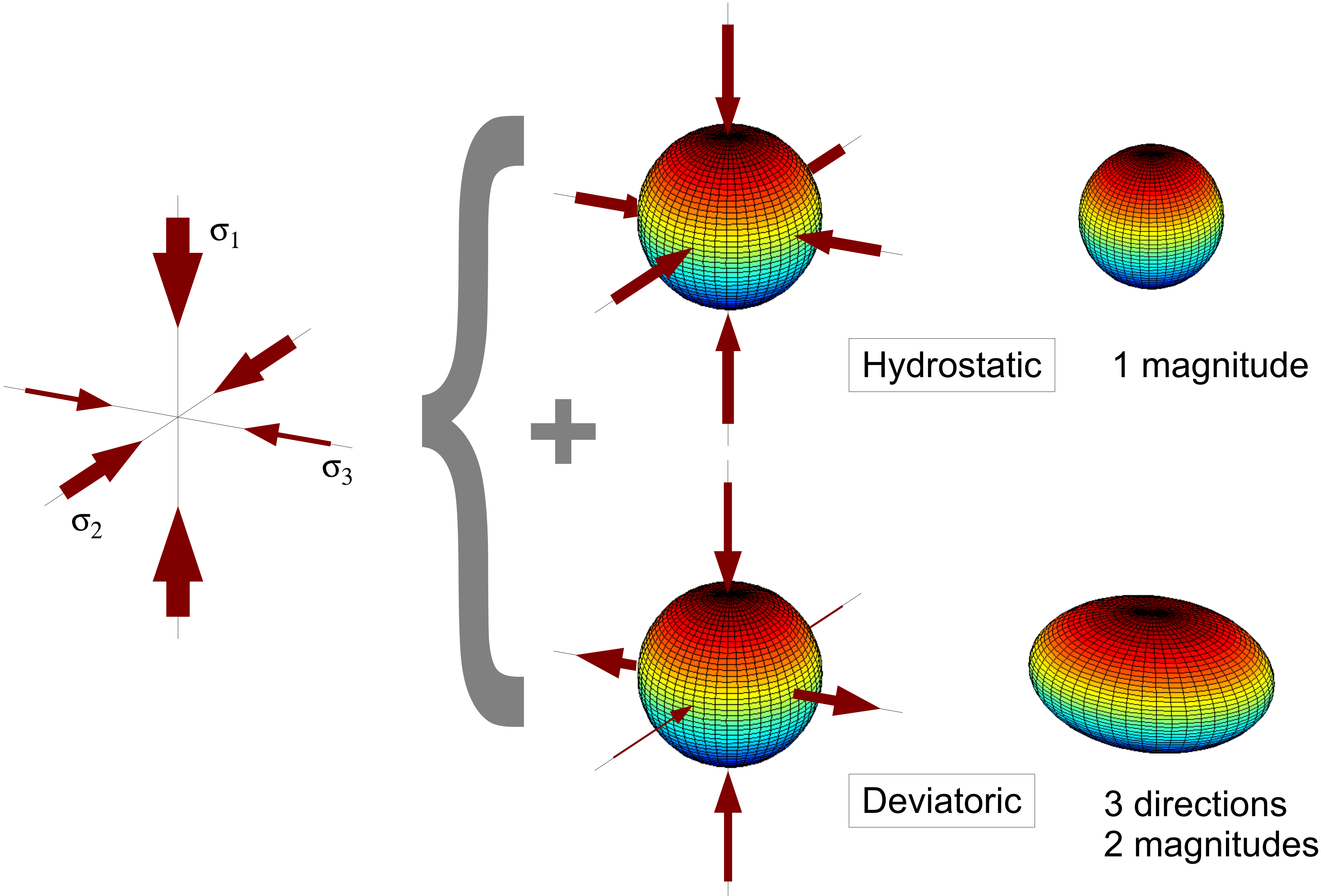
Stress though, is not a single number (scalar), it's six numbers (symmetric tensor).



Stress though, is not a single number (scalar), it's six numbers (symmetric tensor).



Stress though, is not a single number (scalar), it's six numbers (symmetric tensor).

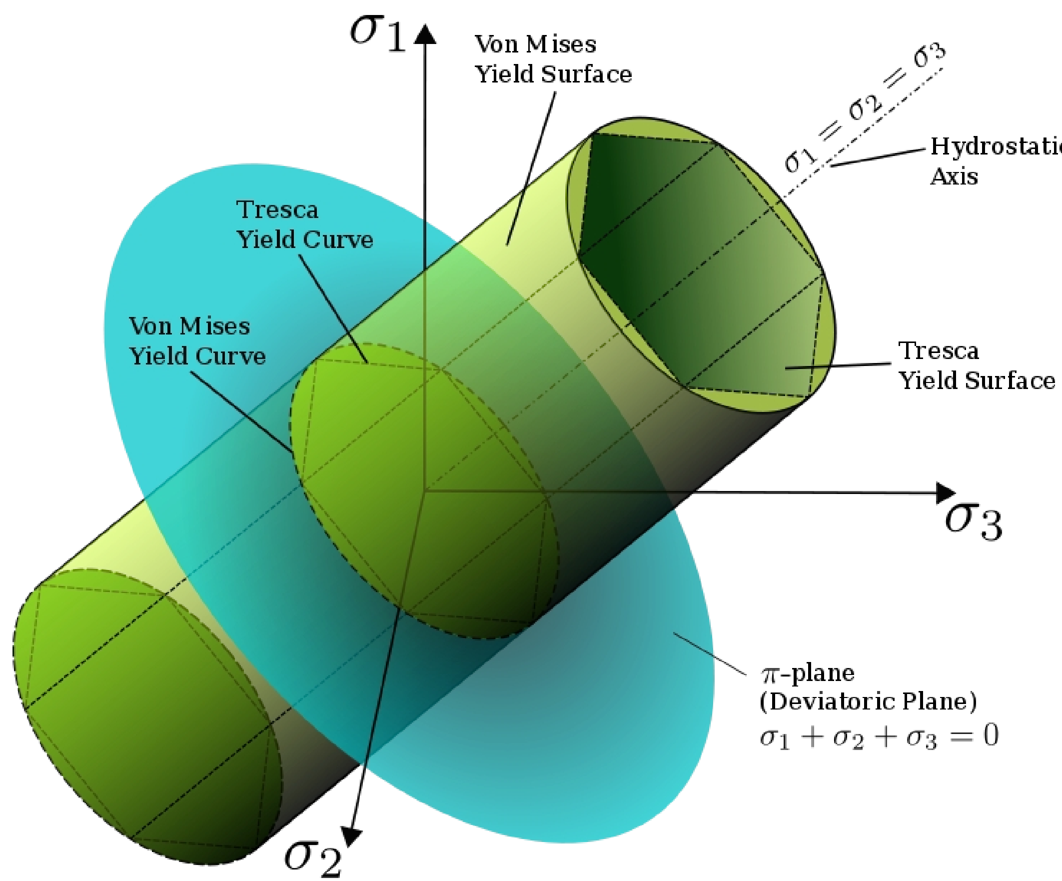


A failure criterion (or envelope) is the particular case of a yield surface for a brittle material. A yield surface can be interpreted as the deviatoric stress state that governs failure/yield/flow (neglecting dependence on hydrostatic stress).

Brittle → Failure

Plastic → Yield

Viscoplastic → Flow



There are 2 main yield or failure criteria:-

Maximum Shear Stress (Tresca)

$$\sigma_1 - \sigma_3 = C$$

Mohr-Coulomb failure envelope

-

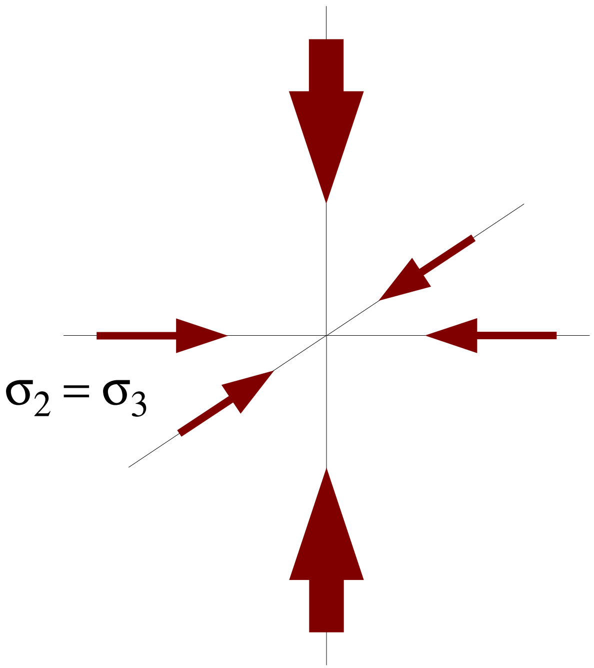
Maximum Distortion Energy (Maxwell, von Mises, Hencky)

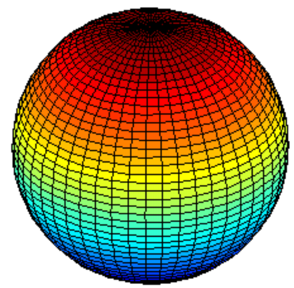
$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{\frac{1}{2}} = C$$

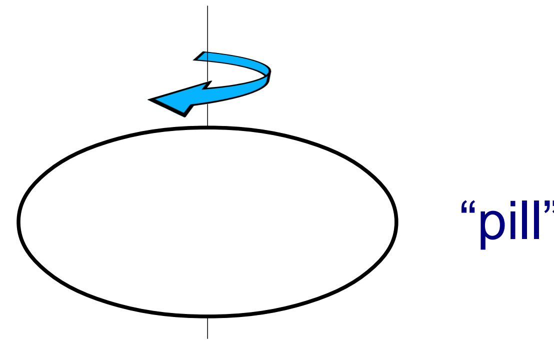
Mogi-Coulomb failure enevelope

However, most of the time, it's impossible for us to tell the difference.

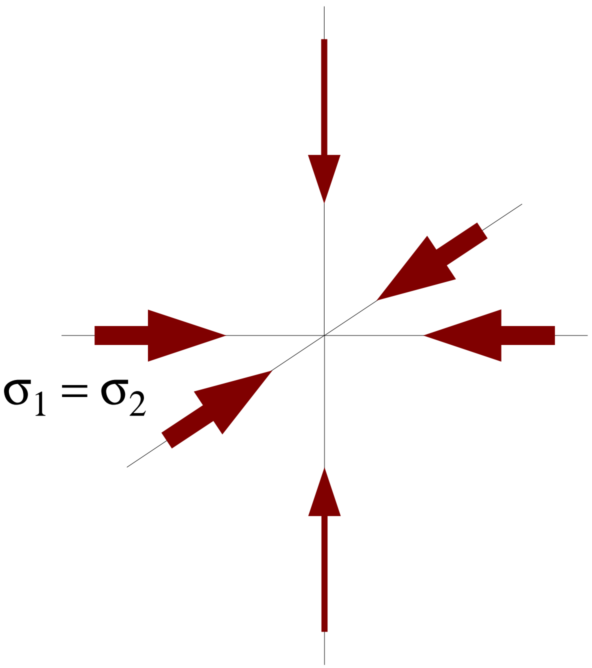
When we measure how materials respond to stress in the laboratory, we keep it cheap and simple, we make two of the principal stresses equal.

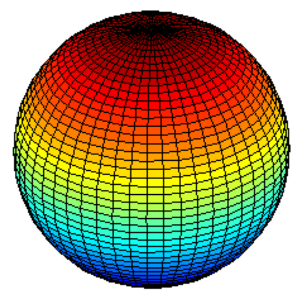


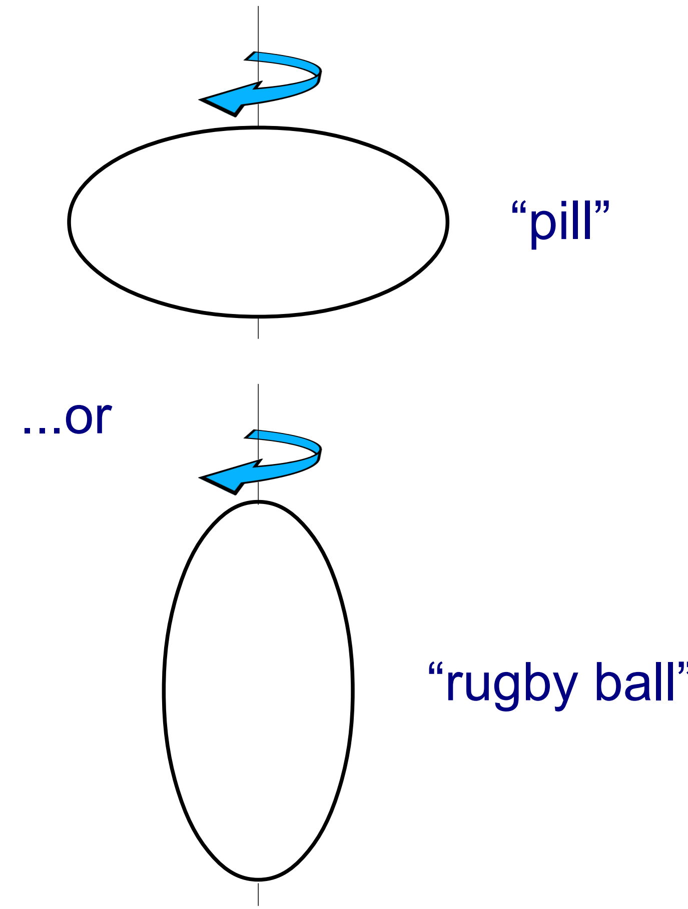




When we measure how materials respond to stress in the laboratory, we keep it cheap and simple, we make two of the principal stresses equal.

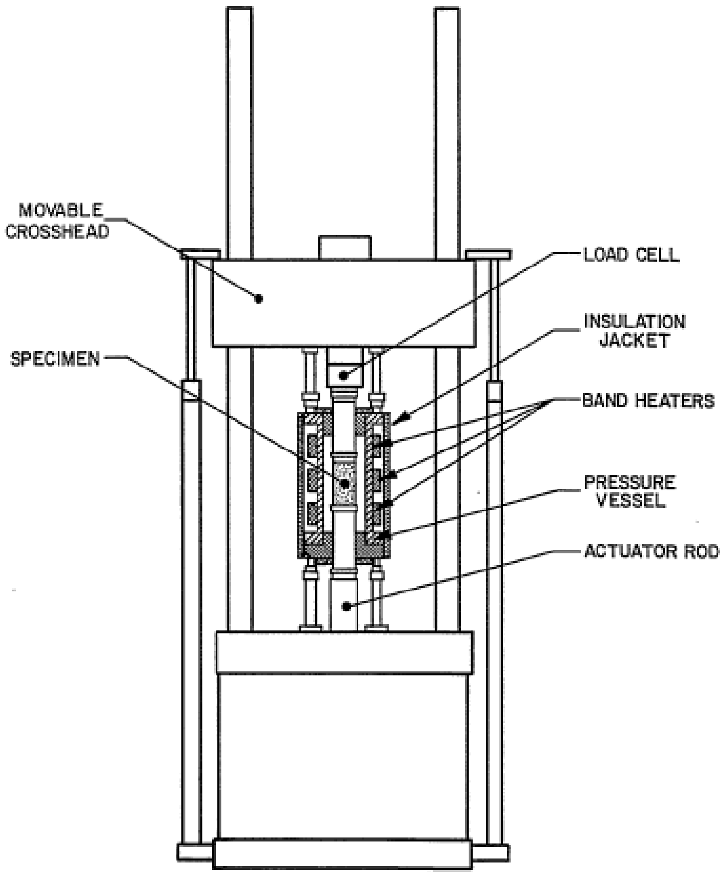


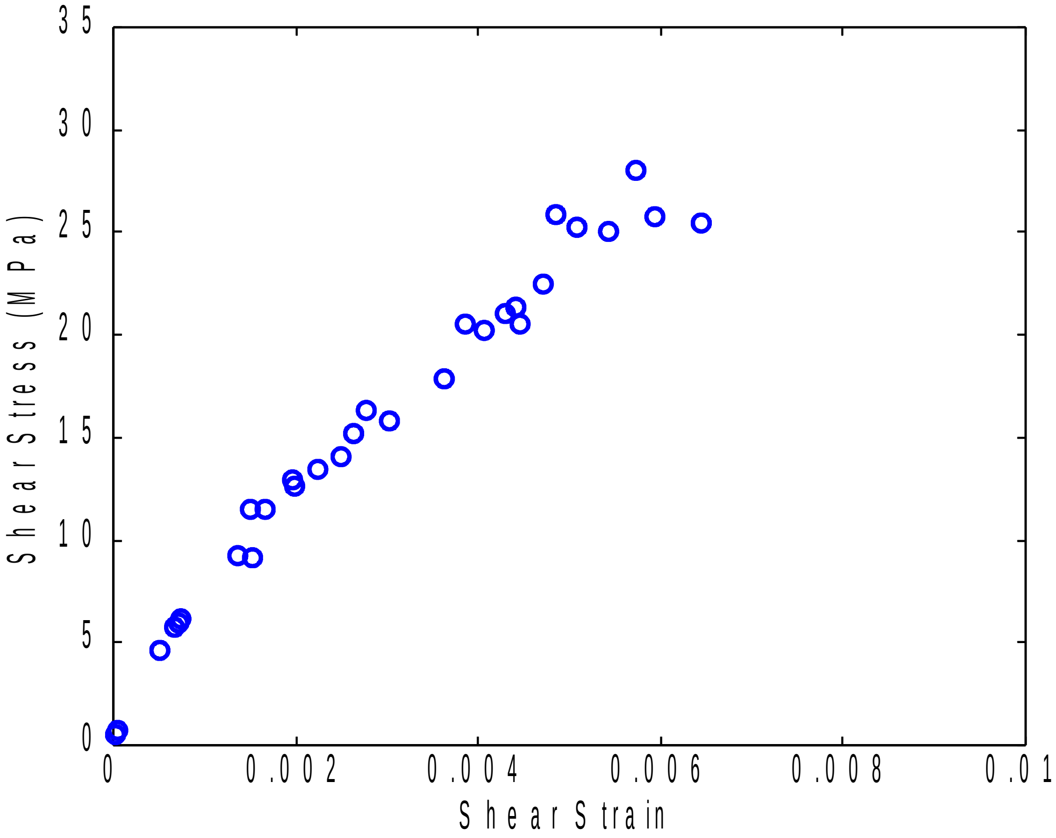




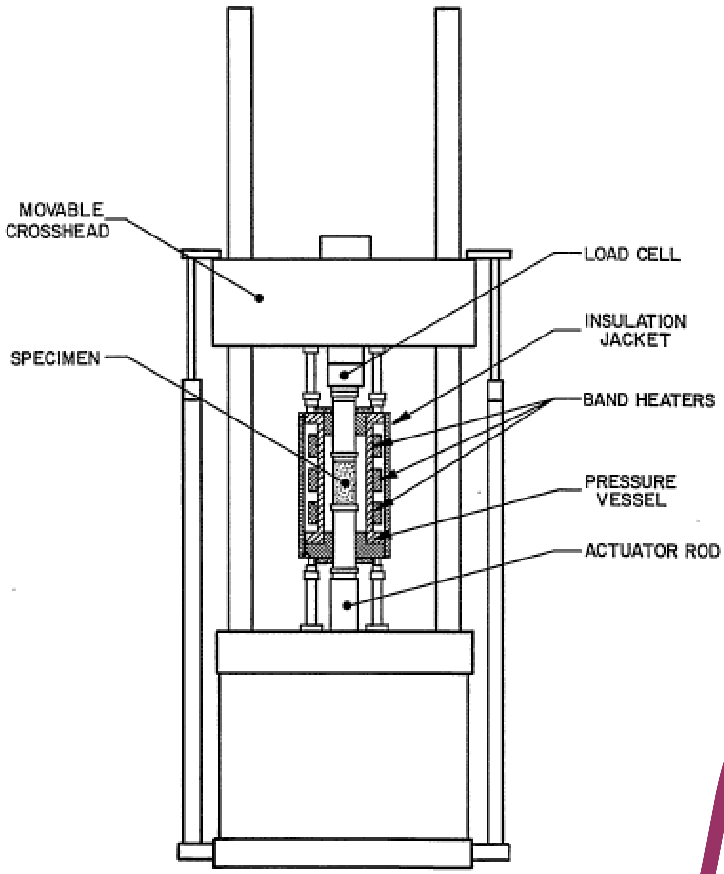
Now the shear stress magnitude can be defined by a single number, but the two shear stress measures are now functionally equivalent and can't be differentiated.

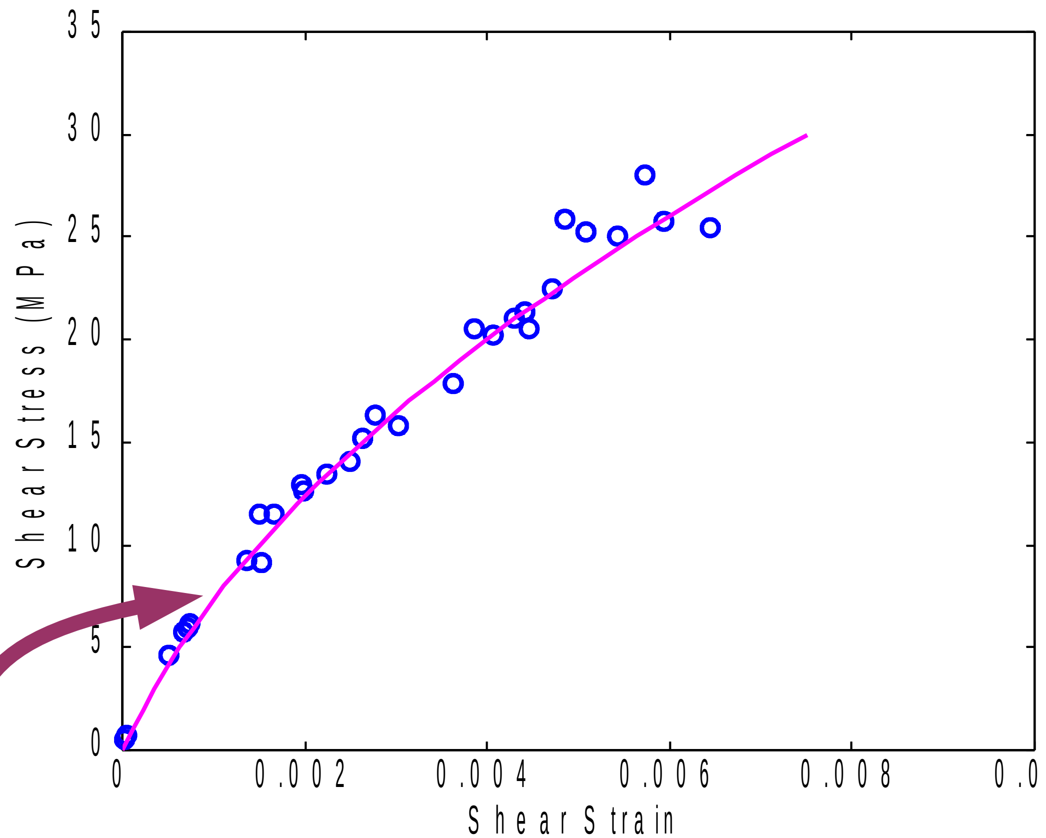
This makes it straightforward to perform deformation experiments and express their results in terms of a single shear stress 'measure'.





This makes it straightforward to perform deformation experiments and express their results in terms of a single shear stress 'measure'.

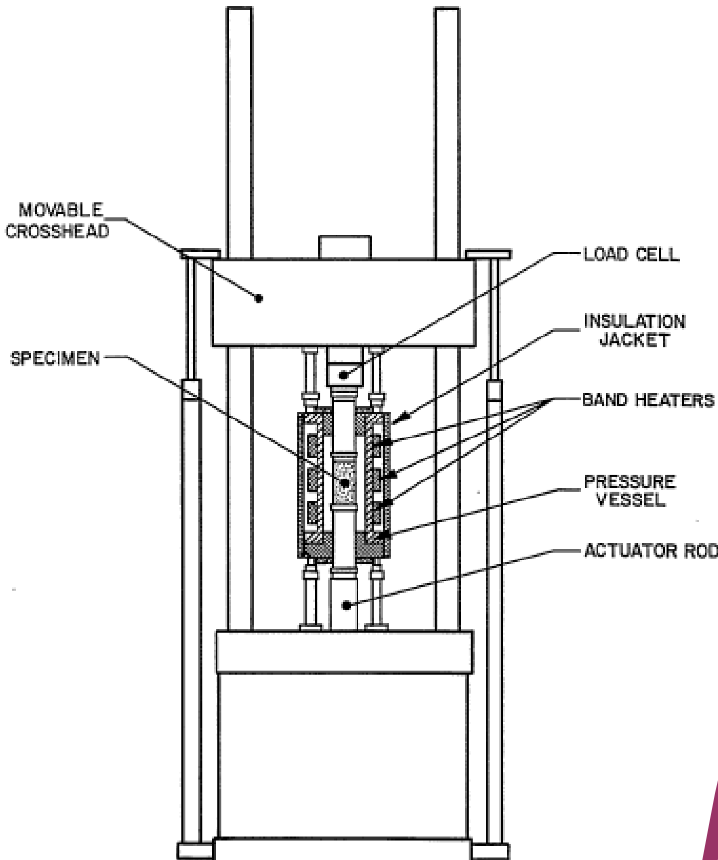


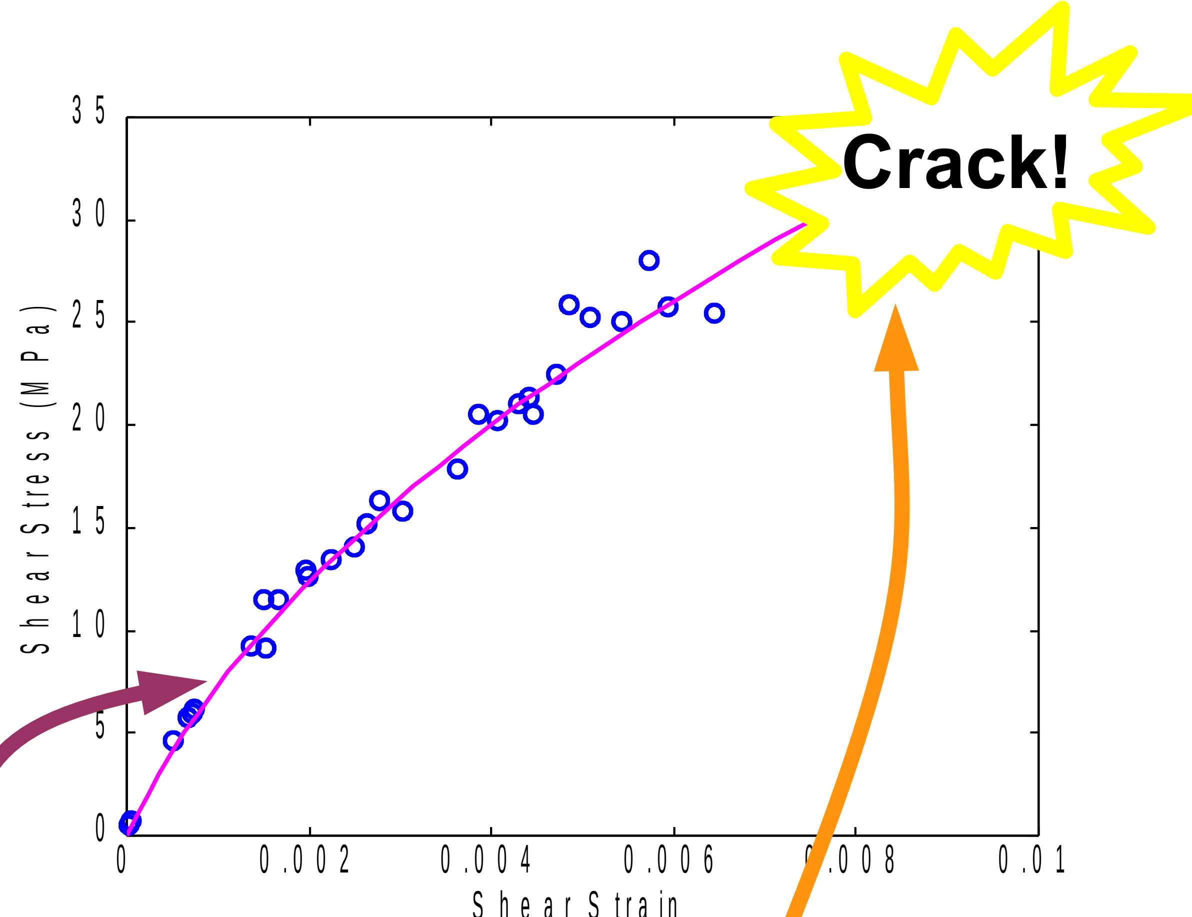


$$\varepsilon = \frac{\tau}{G} + C_0 \tau^2$$

constitutive laws

This makes it straightforward to perform deformation experiments and express their results in terms of a single shear stress 'measure'.





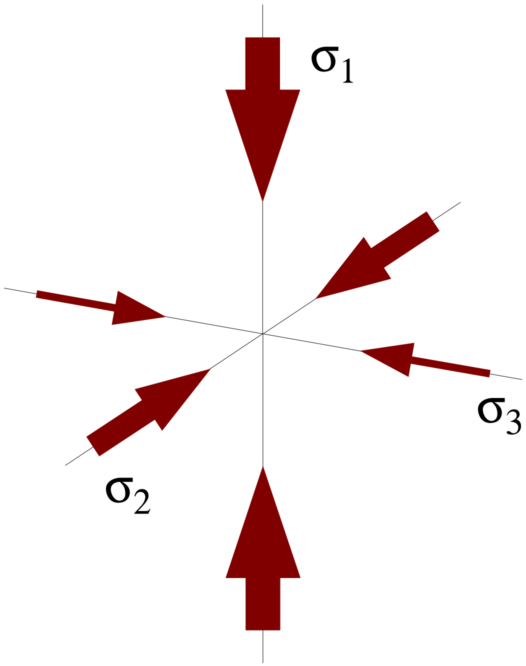
$$\varepsilon = \frac{\tau}{G} + C_0 \tau^2$$

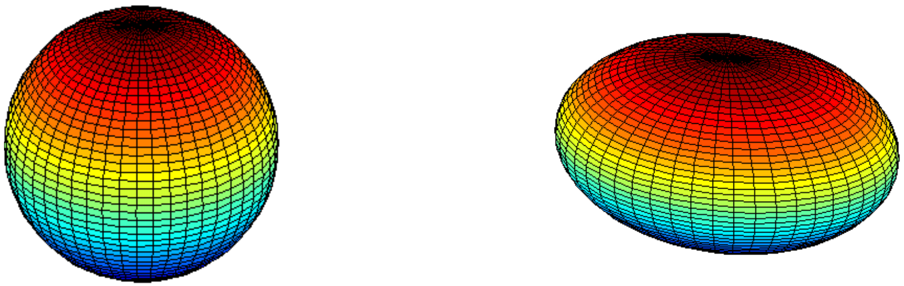
constitutive laws

$$\tau \geq \mu \sigma_n + S_0$$

failure criteria

Sadly, real life is more complex, all three principal stresses are usually different.



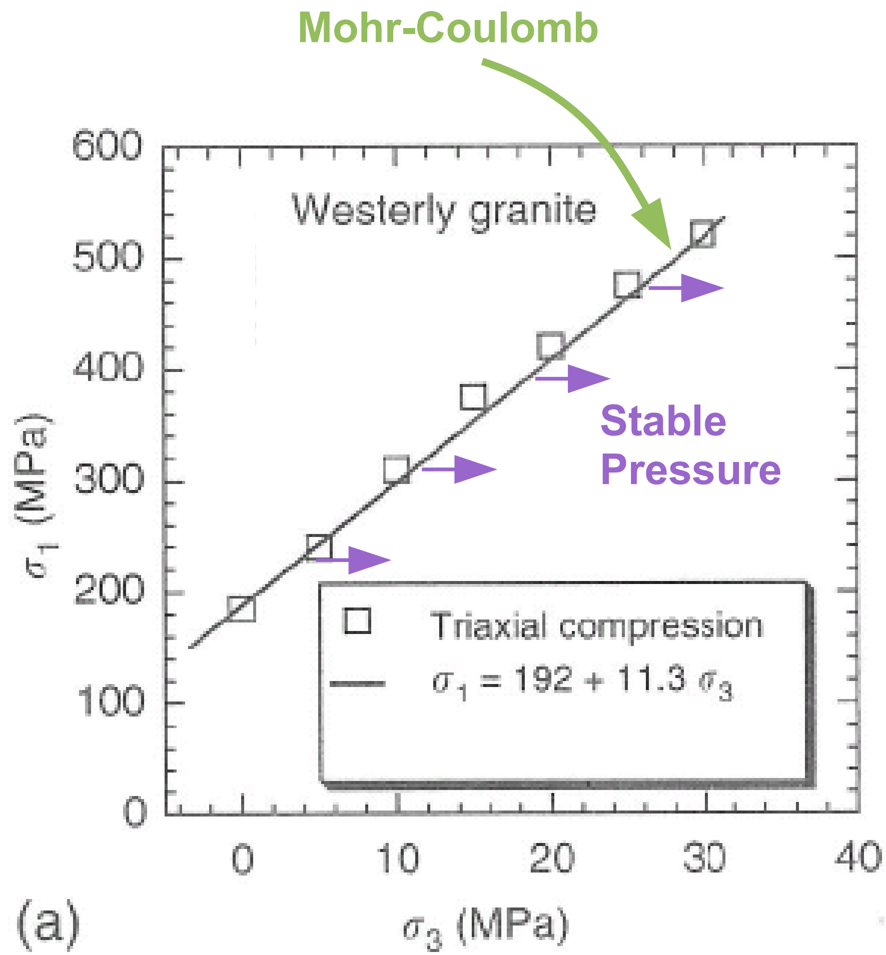


In these cases the shear stress magnitude requires two independent numbers to define it and the two shear stress measures of interest, diverge.

In fact, when all three principal stresses are unequal, the number of valid but functionally unique shear stress measures becomes infinite!

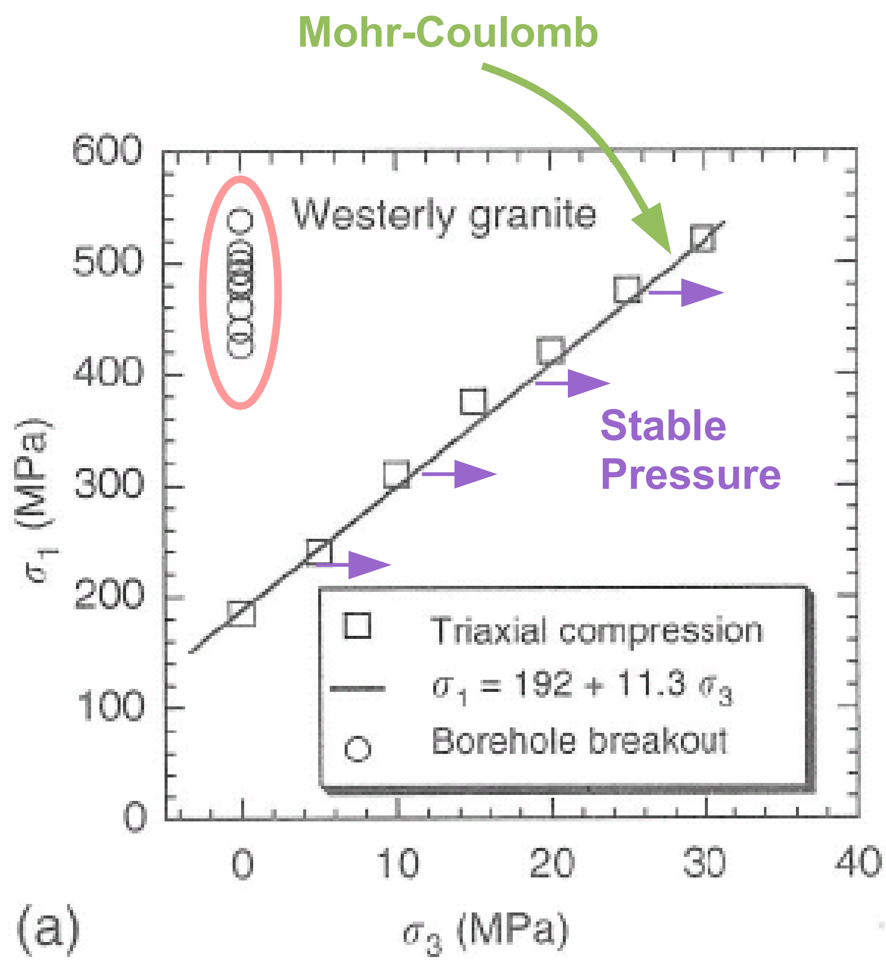
$$\left|\left\{ \tau_m \text{ is a valid shear stress measure},\ \tau_m \text{ is functionally unique} \right\}\right| \rightarrow \infty$$

Does the choice of shear stress measure / failure criterion make much of a difference?



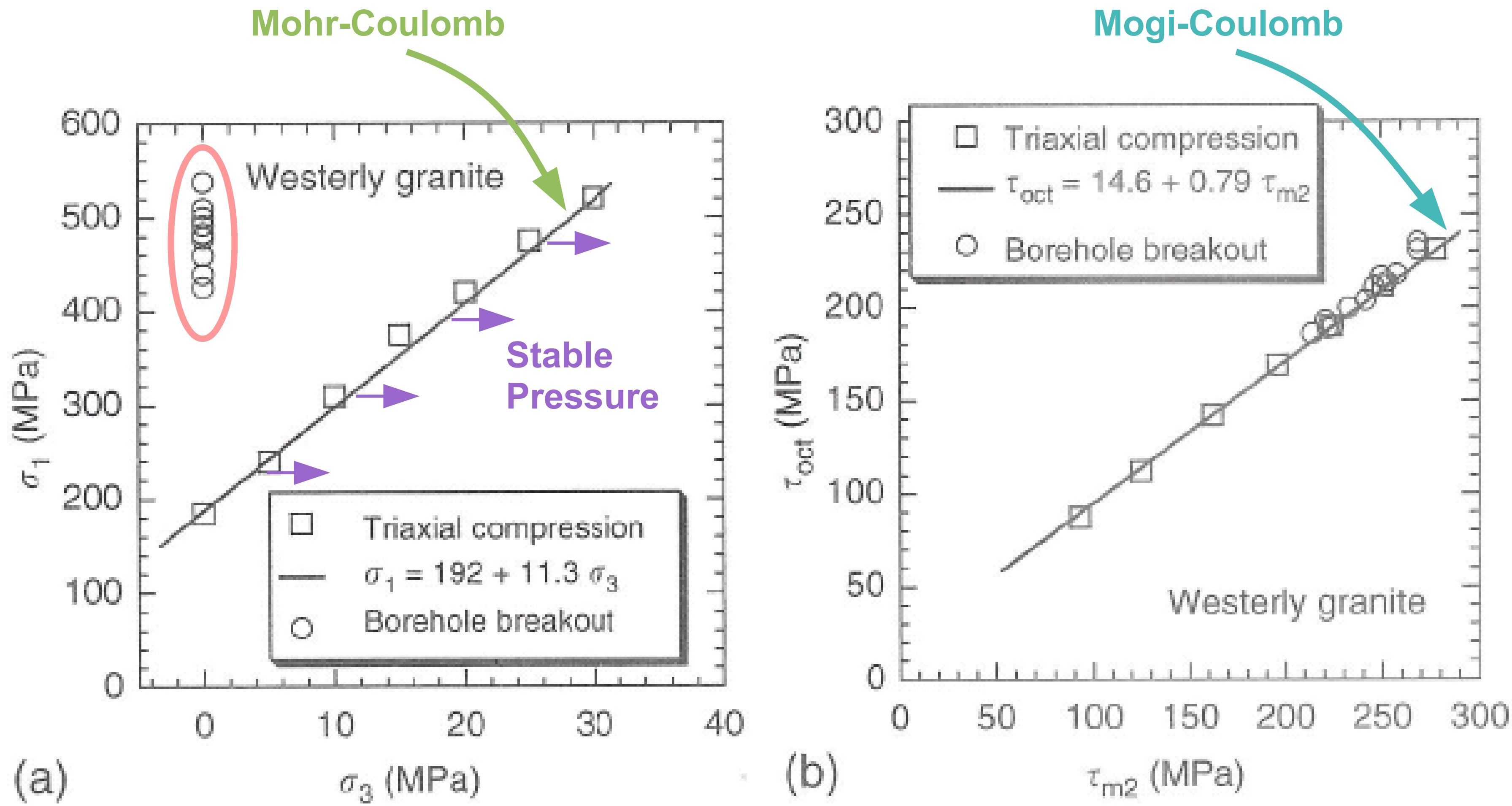
(a)

Does the choice of shear stress measure / failure criterion make much of a difference?



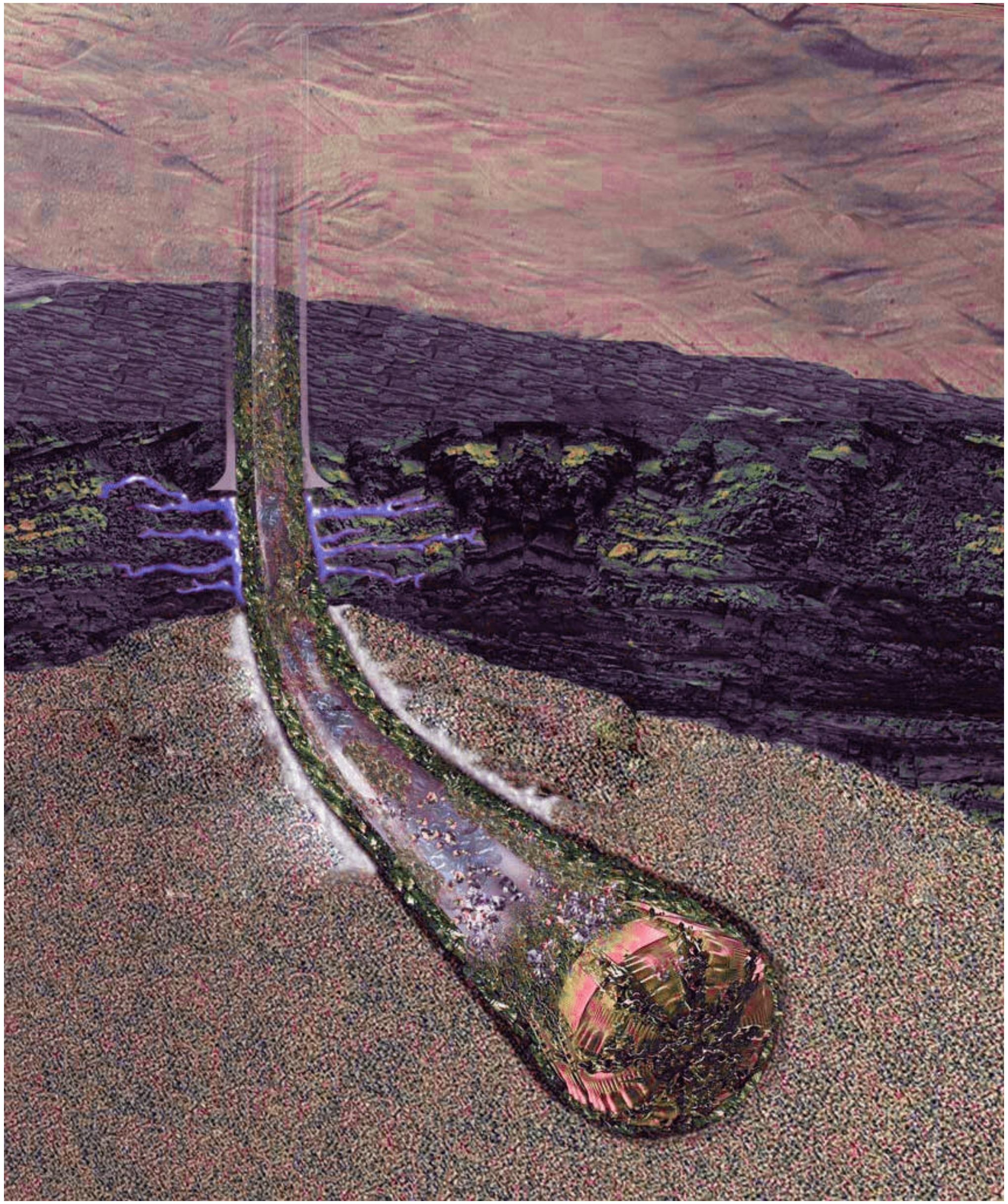
(a)

Does the choice of shear stress measure / failure criterion make much of a difference?



So yes, it can really make a difference for a number of high risk and costly operations in geomechanics, but safety means we have to be conservative – i.e. Mohr-Coulomb, unless we can be sure that Mogi-Coulomb is justified.

...and wellbore stability is of particular interest.

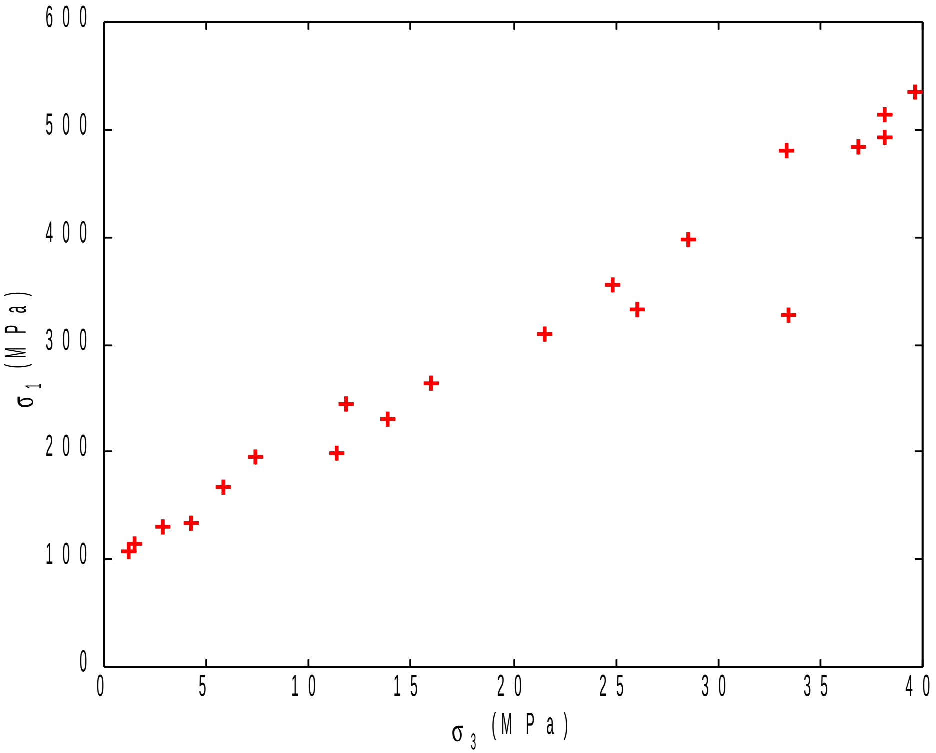


Wellbore stability is a matter of HSE concern as well as raw economics. Well failure can, in the most extreme situations, endanger lives.

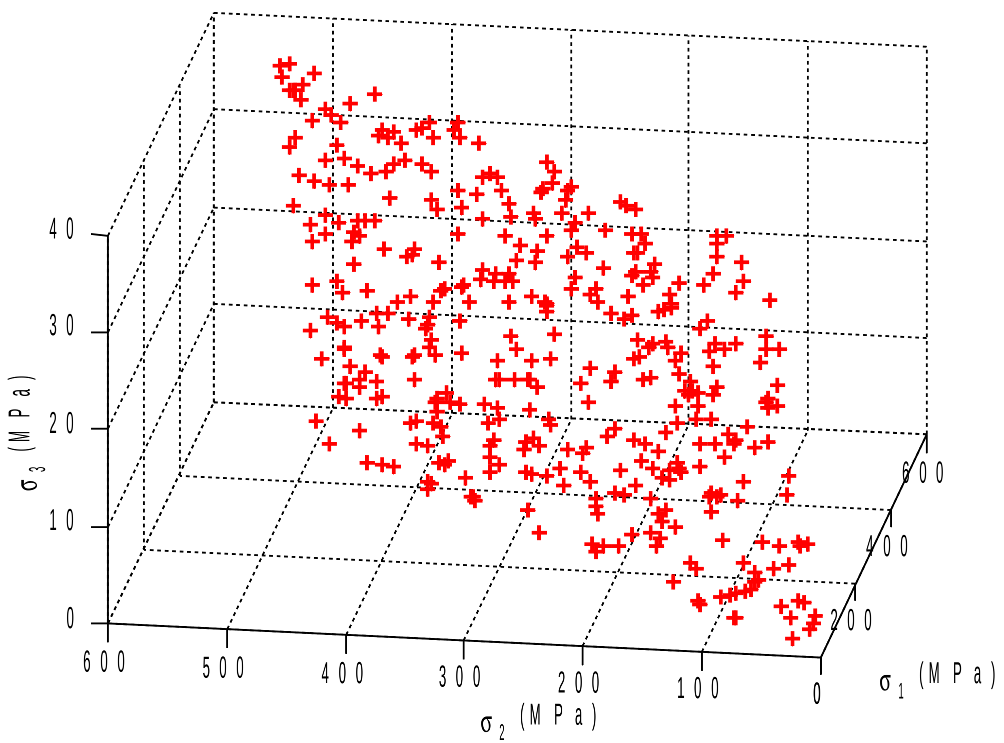
Drilling wells is one of the most expensive things we do as a company, the costliest easily exceed $100 mln.

Understanding and modeling the mechanics of wellbore stability has significant impact on our operations.

But we haven't known how to generalise our 'laboratory triaxial' derived failure relationships to true multiaxial stress states. To properly characterise the multiaxial stress response has required squaring the number of data points.



20 points ~$100,000



400 points ~$2,000,000

The high costs mean that very few materials have been properly characterised for multiaxial behaviour, and that's true for all continuum mechanics disciplines (even mild steel).

There are many references in the literature that state that a distortion energy / principle of least action derived yield surface is only satisfied by a von Mises shear stress measure (octahedral, J2, …), equivalent to a Mogi-Coulomb failure criterion. Hence, by inference, other criteria are not distortion energy based.

$$U_T \;=\; \frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\epsilon} \;=\; \frac{1}{2}\tau_m\epsilon_m \;=\; \frac{1}{6}\tilde{\tau}_i\tilde{\epsilon}_i$$

$\tau_m,\ \epsilon_m$ – shear stress and strain measures

$\tilde{\tau}_i\ ,\ \tilde{\epsilon}_i$ – principal stress and strain deviators, e.g. $\tilde{\tau}_1 = \sigma_1 - \sigma_2$

But this is not actually true, it implicitly assumes a linear relationship between stress and strain. A general distortion energy / principle of least action derived yield surface is fully satisfied by:-

$$\tau_m = \left[(\sigma_1-\sigma_2)^p + (\sigma_2-\sigma_3)^p + (\sigma_1-\sigma_3)^p\right]^{\frac{1}{p}}$$

i.e. the *L-p* norm of the principal stress deviators.

For $p \rightarrow \infty$

$$\tau_m \rightarrow \sigma_1 - \sigma_3$$

i.e. a Tresca or Mohr-Coulomb condition.

For $p = 2$

$$\tau_m = \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{\frac{1}{2}}$$

i.e. a von Mises or Mogi-Coulomb condition.

Hence, both conditions are consistent with a distortion energy based criterion, as are intermediate values of p.

This also provides an explanation for the seemingly inconsistent results for multiaxial flow in power law visco-plastic materials, where flow is observed to depend on σ_2, but only weakly (salt, polymers,...).

The problem is more naturally expressed in terms of the principal deviators rather than the more usual Haigh-Westergaard invariants. It's based on finding the solution for power law rheologies, but the result is general for any power law constitutive relationship, solid or fluid.

$$\dot{\epsilon}_1-\dot{\epsilon}_2=(s_1-s_2)^n+f_{12}(\boldsymbol{s},n); \quad \dot{\epsilon}_2-\dot{\epsilon}_3=(s_2-s_3)^n+f_{23}(\boldsymbol{s},n); \quad \dot{\epsilon}_1-\dot{\epsilon}_3=(s_1-s_3)^n+f_{13}(\boldsymbol{s},n)$$

$$\left[N.B. \quad f_{12}=f_{23}=f_{13}=0 \ \text{ if } \ s_1=s_2, \ \ s_2=s_3, \ \ s_1=s_3, \ \text{ or } \ n=1\right]$$

As $\mathrm{Tr}(\dot{\boldsymbol{\epsilon}})=\dot{\epsilon}_1+\dot{\epsilon}_2+\dot{\epsilon}_3=0$, it follows that:

$$\begin{aligned}
\dot{\epsilon}_1-\dot{\epsilon}_2&=\left[2(s_1-s_2)^n+(s_1-s_3)^n-(s_2-s_3)^n+2f_{12}-f_{23}+f_{13}\right]/3\\
\dot{\epsilon}_2-\dot{\epsilon}_3&=\left[2(s_2-s_3)^n+(s_1-s_3)^n-(s_1-s_2)^n+2f_{23}-f_{12}+f_{13}\right]/3\\
\dot{\epsilon}_1-\dot{\epsilon}_3&=\left[2(s_1-s_3)^n+(s_1-s_2)^n+(s_2-s_3)^n+2f_{13}+f_{12}+f_{23}\right]/3
\end{aligned}$$

Hence

$$\begin{aligned}
f_{12}&=\left[-(s_1-s_2)^n+(s_1-s_3)^n-(s_2-s_3)^n\right]/3\\
f_{23}&=\left[-(s_2-s_3)^n+(s_1-s_3)^n-(s_1-s_2)^n\right]/3\\
f_{13}&=\left[-(s_1-s_3)^n+(s_1-s_2)^n+(s_2-s_3)^n\right]/3
\end{aligned}$$

So that gives us the tensor form of the stress-strain relationship for power law rheological materials, and it can also be shown that it's true for shear strain in solids, (shear stress doesn't cause volume strain to first order in solids).

Now for the transformation to a scalar shear stress measure.

Thermodynamics requires that: $\sigma_{ij}\, d\epsilon_{ij} = \sigma_m \dfrac{\partial \epsilon_m}{\partial \epsilon_{ij}}\, d\epsilon_{ij}$

i.e. the shear stress measure and conjugate shear strain measure are essentially proxies for the shear energy

This leads directly to: $\epsilon_{ij} = \dfrac{\partial \sigma_m}{\partial \sigma_{ij}}\, \epsilon_m$ or the time derivative form: $\dot{\epsilon}_{ij} = \dfrac{\partial \sigma_m}{\partial \sigma_{ij}}\, \dot{\epsilon}_m$

which transforms the tensor form stress-strain relationship for shear in power law materials to:

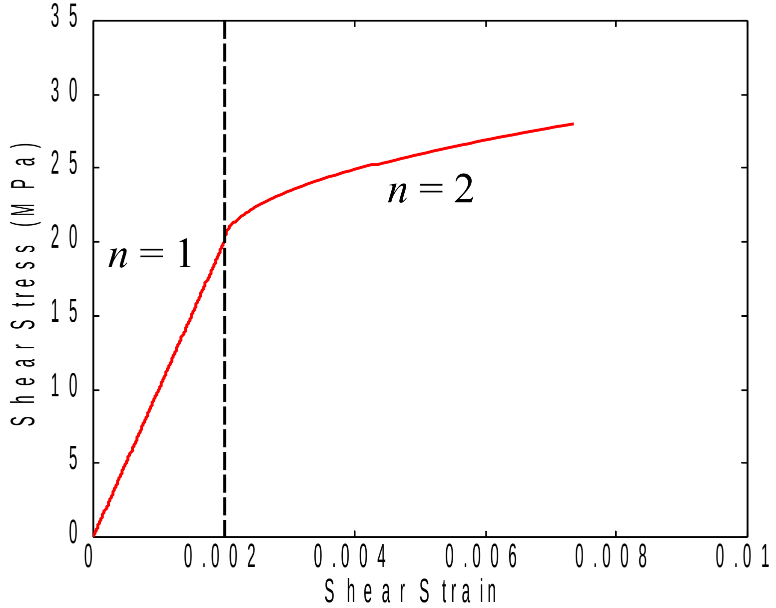
$$\boxed{\sigma_m = \left[(s_1 - s_2)^{n+1} + (s_2 - s_3)^{n+1} + (s_1 - s_3)^{n+1} \right]^{\frac{1}{n+1}}}$$

The 'correct' shear stress measure, depends on the constitutive properties (the stress-strain relationship) of the material in question.

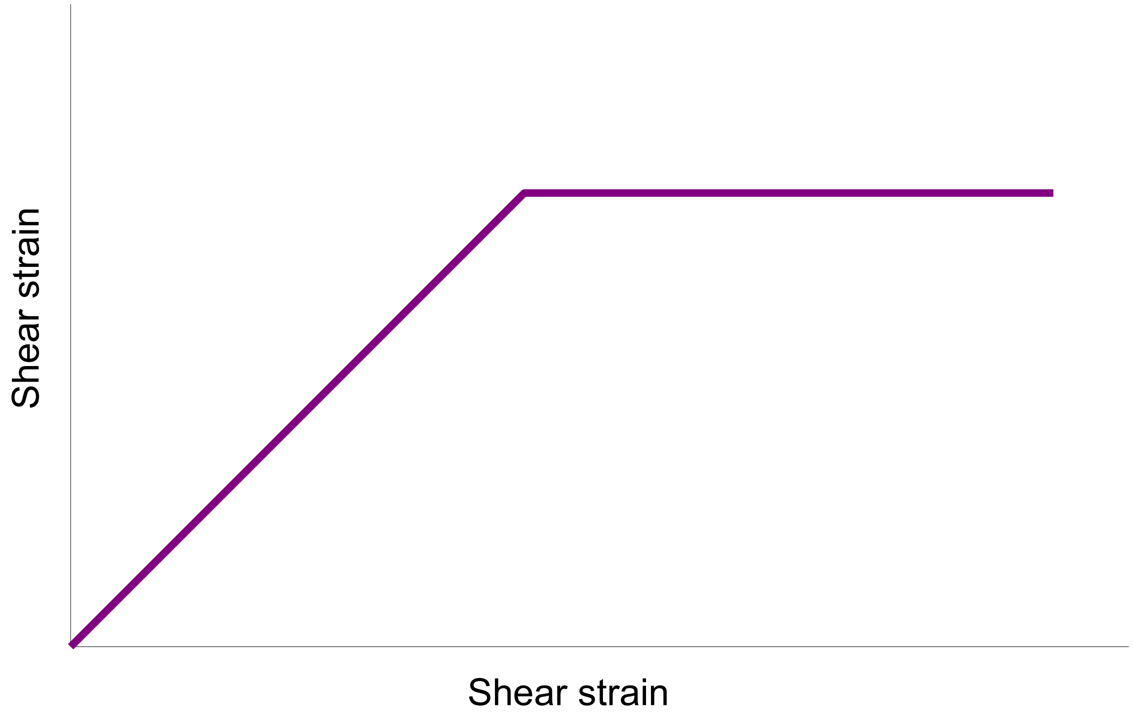
For a material where the shear stress – shear strain relationship is power-law '*n*':

$$\tau_m = \left[(\sigma_1 - \sigma_2)^{n+1} + (\sigma_2 - \sigma_3)^{n+1} + (\sigma_1 - \sigma_3)^{n+1} \right]^{\frac{1}{n+1}}$$

And we can always factorise a constitutive relationship using power-law decomposition...

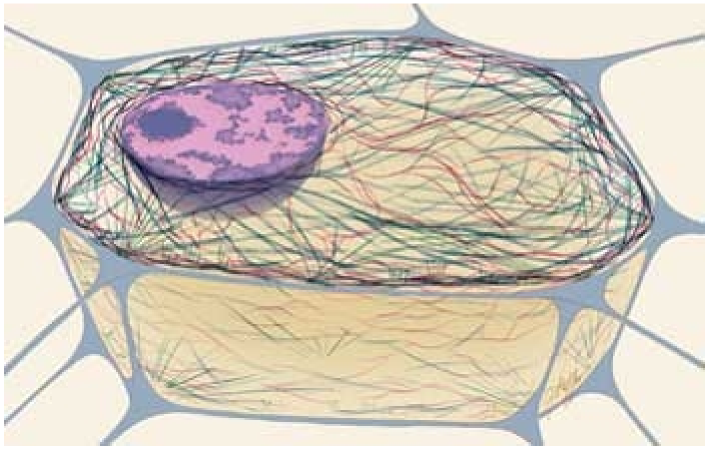


However, the *L-p* norms converge very rapidly for increasing values of *p*, so for practical purposes it is simplest to apply a Mogi-Coulomb / von Mises yield condition to materials that exhibit linear elastic behaviour until yield / failure.

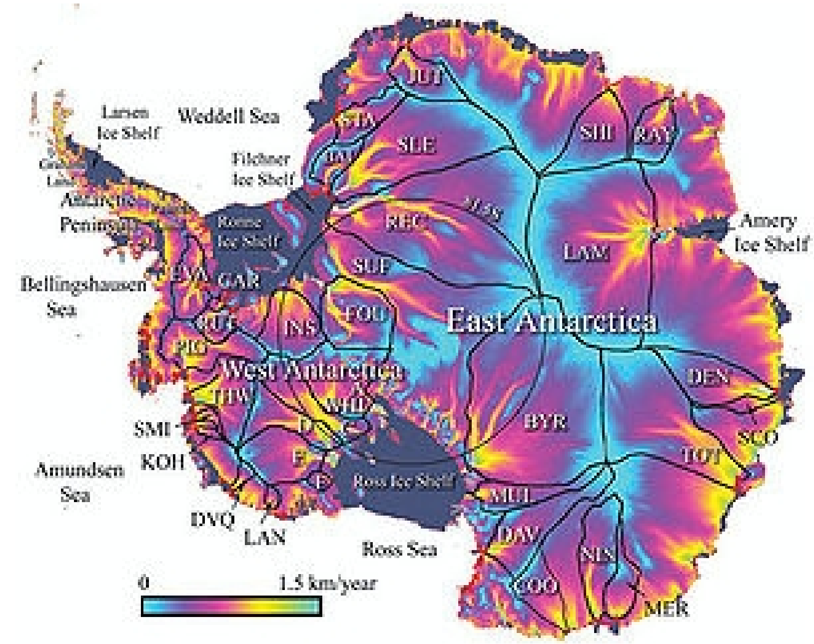


For materials that exhibit disproportionate shear strain with increasing shear stress, a Mohr-Coulomb / Tresca yield condition is more appropriate.

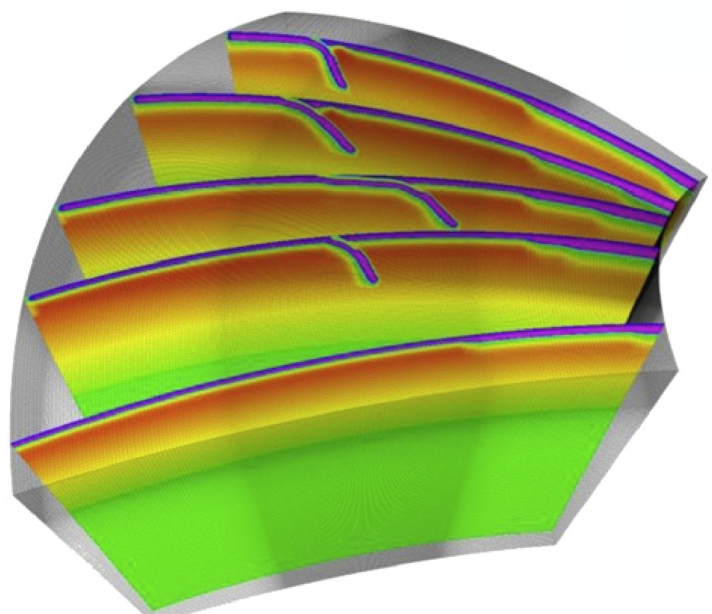
Wide range of applications – some less familiar...



Biology

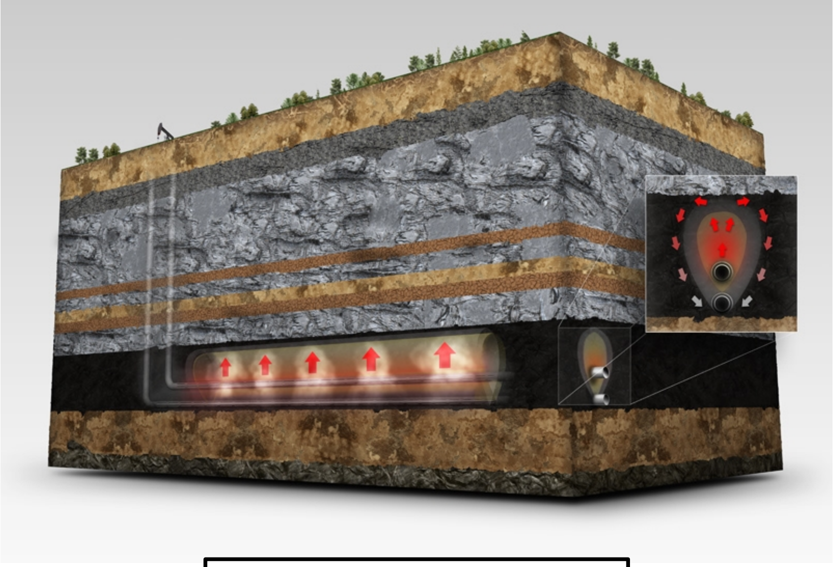


Glaciers

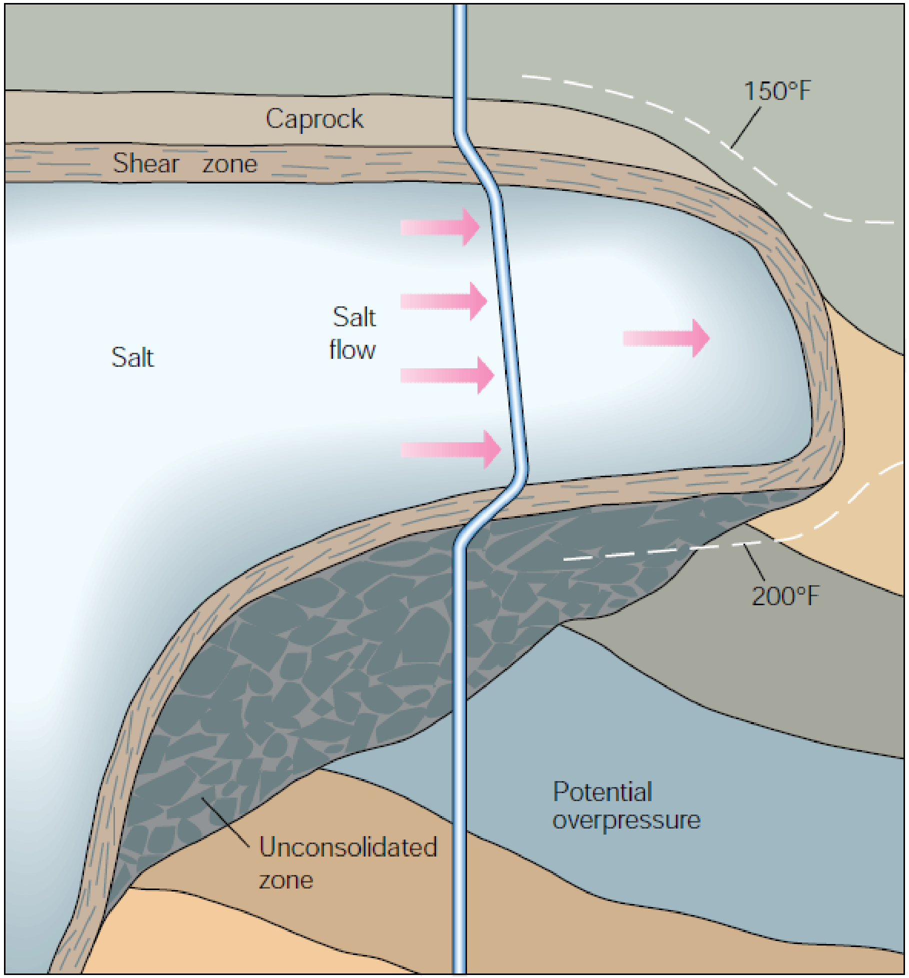


Tectonics

...some more familiar...



Tars & Heavy Oils



Salt Modeling

The correct shear stress measure, required to generalise laboratory 'triaxial' data to true multiaxial conditions, depends on the constitutive properties of the material itself.

$$\boxed{\tau_m = \left[(\sigma_1 - \sigma_2)^{n+1} + (\sigma_2 - \sigma_3)^{n+1} + (\sigma_1 - \sigma_3)^{n+1} \right]^{\frac{1}{n+1}}}$$

Failure conditions should take account of the 'correct' shear stress measure, e.g. Mogi-Coulomb for linear brittle materials (… and Mohr-Coulomb otherwise).