recent progress in
WAVE DIGITAL AUDIO EFFECTS

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THANKS

• DAFx organizing committee

• @ CCRMA
  • Vaibhav Nangia & Jonathan Abel
  • Ross Dunkel & Max Rest & Michael Olsen
  • François Germain

• @ Politecnico di Milano
  • Alberto Bernardini & Augusto Sarti
Musicians like vintage stuff.
Musicians like vintage stuff.
**TWO APPROACHES TO MODELING VINTAGE GEAR**

Nonlinear System Identification ("black box")

- No knowledge of circuit required
- Run test signals to characterize model
- Non-parametric model
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Physical Modeling ("white box")
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- No need to characterize system
- Parametric model
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Physical Modeling (“white box”)

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INTRODUCTION

1. tutorial review of WDF principles
2. recent theoretical progress in WDFs
3. WDF software overview and demo
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1. Everything You Always Wanted to Know About WDFs* (*But Were Afraid to Ask)
2. recent theoretical progress in WDFs
3. WDF software overview and demo
I N T R O D U C T I O N

1. Everything You Always Wanted to Know About WDFs* (*But Were Afraid to Ask)
2. research by DAFx folks (and new research intro by Kurt et al. @ CCRMA)
3. WDF software overview and demo
INTRODUCTION

1. Everything You Always Wanted to Know About WDFs* (*But Were Afraid to Ask)
2. research by DAFx folks
   (and new research intro by Kurt et al. @ CCRMA)
3. “Please, no more math!!!”
   “Just show us how to code it up…”
**Wave Digital Filter History**

- 1989–present: nonlinear theory
- 1996–present: virtual analog / physical modeling applications
Wave Digital Filter Basics

WDF approach involves:

- introduction of free parameter (port resistance) at each port:
  \[ R_n > 0 \], for each port \( n \)

- introduction of wave variables:
  \[ a_n = v_n + R_n i_n \]
  \[ b_n = v_n - R_n i_n \]

- discretization of reactive elements (capacitors, inductors) using the Bilinear transformation:
  \[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}} \], \( c = 2/T \) (typically)

- scattering at impedance mismatches
- resolve delay-free loops by tuning port impedances
**Wave Digital Filter Basics**

closely related to **Digital Waveguides (DWG)**, where:

- wave propagation characterized by *physical* transmission impedance
  \[
  R_n > 0 \text{, for each port } n
  \]

- introduction of wave variables:
  \[
  v_n^+ = (1/2)v_n + (R_n/2)i_n \\
  v_n^- = (1/2)v_n - (R_n/2)i_n
  \]

- discretization of lumped impedances (bridge, nut, etc.)
  using the Bilinear transformation:
  \[
  s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \text{ (typically)}
  \]

- scattering at impedance mismatches
- propagation delay decouples elements
WAVE DIGITAL FILTER BASICS

difference between **WDFs** and **DWGs**???

- *abstract* vs. *physical* meaning of port impedances

- slight different in variable definition and notation

- WDFs have an extra layer of realizability issues—they can be considered DWGs with *length-zero* transmission lines

- basic DWG formulation is *distributed*—waves are observed
- basic WDF formulation is *lumped*...why wave variables then?
LUMPED SYSTEMS

“A lumped system is one in which the dependent variables of interest are a function of time alone. In general, this will mean solving a set of ordinary differential equations (ODEs).”
LUMPED SYSTEMS

“A lumped system is one in which the dependent variables of interest are a function of time alone. In general, this will mean solving a set of ordinary differential equations (ODEs).”

...as opposed to distributed systems where dependent variables are also a function of space (PDEs)....

**LUMPED ELEMENTS** (electrical)

Graphical representation of the three basic elements in electrical systems.

- $r_E = \text{electrical resistance}$
- $L = \text{inductance}$
- $C_E = \text{electrical capacitance}$
**LUMPED ELEMENTS** (acoustical)

Graphical representation of the three basic elements in electrical and acoustical systems.

- \( r_E \) = electrical resistance
- \( r_A \) = acoustical resistance
- \( L \) = inductance
- \( M \) = inertance
- \( C_E \) = electrical capacitance
- \( C_A \) = acoustical capacitance

---

LUMPED ELEMENTS (mechanical rectilinear)

Graphical representation of the three basic elements in electrical, mechanical, and acoustical systems.

- **Electrical**
  - $r_E = \text{electrical resistance}$
  - $L = \text{inductance}$
  - $C_E = \text{electrical capacitance}$

- **Acoustical**
  - $r_A = \text{acoustical resistance}$
  - $M = \text{inertance}$
  - $C_A = \text{acoustical capacitance}$

- **Mechanical**
  - $r_M = \text{mechanical rectilinear resistance}$
  - $m = \text{mass}$
  - $C_M = \text{compliance}$

LUMPED ELEMENTS (mechanical rotational)

Graphical representation of the three basic elements in electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

- $r_E = \text{electrical resistance}$
- $L = \text{inductance}$
- $C_E = \text{electrical capacitance}$

- $r_A = \text{acoustical resistance}$
- $M = \text{inertance}$
- $C_A = \text{acoustical capacitance}$

- $r_M = \text{mechanical rectilineal resistance}$
- $m = \text{mass}$
- $C_M = \text{compliance}$

- $r_R = \text{mechanical rotational resistance}$
- $I = \text{moment of inertia}$
- $C_R = \text{rotational compliance}$

LUMPED ELEMENTS (equivalence across domains)

![Diagram of Lumped Elements](image)

Graphical representation of the three basic elements in electrical, mechanical rectilinear, mechanical rotational and acoustical systems.

<table>
<thead>
<tr>
<th></th>
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- $r_E = \text{electrical resistance}$
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- $m = \text{mass}$
- $I = \text{moment of inertia}$
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Graphical representation of the three basic elements in electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

A LUMPED SYSTEM (mechanical rotational)

Electrical, mechanical rectilineal, mechanical rotational and acoustical systems of one degree of freedom and the current, velocity, angular velocity and volume current response characteristics.

\[
\begin{align*}
I(s) & = \frac{C_E s}{LC_E s^2 + C_E r_E s + 1} \\
E(s) & = \frac{C_E s}{LC_E s^2 + C_E r_E s + 1} \\
\dot{X}(s) & = \frac{C_A s}{MC_A s^2 + C_A r_A s + 1} \\
F_M(s) & = \frac{C_M s}{mC_M s^2 + C_M r_M s + 1} \\
\dot{\Phi}(s) & = \frac{C_R s}{IC_M s^2 + C_R r_R s + 1} \\
F_R(s) & = \frac{C_R s}{IC_M s^2 + C_R r_R s + 1}
\end{align*}
\]
NETWORK THEORY (port definition)

Ports have:
NETWORK THEORY (port definition)

Ports have:

- two terminals, + and –
NETWORK THEORY (port definition)

Ports have:

- two terminals, + and –
- a voltage $v$ across the terminals
NETWORK THEORY (port definition)

Ports have:

- two terminals, + and –
- a voltage $v$ across the terminals
- a current $i$ heading into the + terminal and out of the – terminal
NETWORK THEORY (port definition)

Ports have:

- two terminals, + and –
- a voltage \( v \) across the terminals
- a current \( i \) heading into the + terminal and out of the – terminal
- a port resistance \( R_p \) that characterizes the port (wave domain)
**NETWORK THEORY** (port definition)

Ports have:

- two terminals, + and –
- a voltage $v$ across the terminals
- a current $i$ heading into the + terminal and out of the – terminal
- a port resistance $R_p$ that characterizes the port (wave domain)

linear **One-Ports** include:

NETWORK THEORY (n-ports)

- connected ports have equal port resistance
- 2-ports (e.g. transformers, parallel/series connections)
- 3-ports (e.g. parallel/series connections)
- 4+ ports, etc.
- mismatches of port resistance and topological aspects handled by “adaptors”, where “scattering” of wave variables occurs
DISCRETIZATION

replace all continuous-time derivatives $s$ on Laplace plane with discrete-time approximations (in delays $z^{-1}$)

\begin{align*}
\text{forward Euler} & : \quad s \leftarrow \frac{1 - z^{-1}}{Tz^{-1}} \\
\text{backward Euler} & : \quad s \leftarrow \frac{1 - z^{-1}}{T} \\
\text{bilinear transform} & : \quad s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = \frac{2}{T} \text{ (typically)}
\end{align*}

**DISCRETIZATION**

*replace all continuous-time derivatives* \( s \) *on Laplace plane with discrete-time approximations (in delays* \( z^{-1} \))

- **forward Euler**
  \[
  s \leftarrow \frac{1 - z^{-1}}{T z^{-1}}
  \]

- **backward Euler**
  \[
  s \leftarrow \frac{1 - z^{-1}}{T}
  \]

- **bilinear transform**
  \[
  s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \quad \text{(typically)}
  \]

**Discretization**

*replace all continuous-time derivatives* $s$ *on Laplace plane with discrete-time approximations (in delays* $z^{-1}$*)

- **forward Euler**
  
  $s \leftarrow \frac{1 - z^{-1}}{Tz^{-1}}$

- **backward Euler**
  
  $s \leftarrow \frac{1 - z^{-1}}{T}$

- **bilinear transform**
  
  $s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$, $c = 2/T$ *(typically)*

**Delay-Free Loops**

Mutual, instantaneous dependence or “delay-free loop” (implicit)

Delay makes things computable (explicit)

**not OK**

**OK**
DISCRETIZE RC NETWORK (traditional approach)
**DISCRETIZE RC NETWORK** (traditional approach)

\[
\frac{1}{sC} = \frac{E(s) - V_{\text{out}}(s)}{R}
\]

KCL at “+” node

![Diagram of an RC network]
DISCRETIZE RC NETWORK (traditional approach)

\[
\begin{align*}
e(t) & \quad R \\ \text{+} & \quad C \quad v_{\text{out}}(t) \quad \text{+} \\
\end{align*}
\]

\[
\frac{V_{\text{out}}(s)}{1} = \frac{E(s) - V_{\text{out}}(s)}{R}
\]

\[
sRC V_{\text{out}}(s) = E(s) - V_{\text{out}}(s)
\]

\[
V_{\text{out}}(s)(sRC + 1) = E(s)
\]

H(s) = \frac{V_{\text{out}}(s)}{E(s)} = \frac{1}{sRC + 1}

KCL at “+” node

s-plane transfer function
**DISCRETIZE RC NETWORK** (traditional approach)

Let $v_{out}(t)$ generate figures for the Google presentation.

\[ V_{out}(s) = \frac{E(s) - V_{out}(s)}{R} \]

\[ sRC V_{out}(s) = E(s) - V_{out}(s) \]

\[ V_{out}(s)(sRC + 1) = E(s) \]

\[ H(s) = \frac{V_{out}(s)}{E(s)} = \frac{1}{sRC + 1} \]

\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \quad \text{(typically)} \]

DISCRETIZE RC NETWORK (traditional approach)

\[ e(t) \quad R \quad C \quad v_{out}(t) \]

KCL at “+” node

\[ \frac{V_{out}(s)}{sC} = \frac{E(s) - V_{out}(s)}{R} \]

\[ sRC V_{out}(s) = E(s) - V_{out}(s) \]

\[ V_{out}(s)(sRC + 1) = E(s) \]

\[ H(s) = \frac{V_{out}(s)}{E(s)} = \frac{1}{sRC + 1} \]

s - plane transfer function

\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = \frac{2}{T} \] (typically)

\[ \frac{V_{out}(z)}{E(z)} = \frac{1}{\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} RC + 1} \]

DISCREITIZE RC NETWORK (traditional approach)

KCL at “+” node

\[ \frac{V_{\text{out}}(s)}{\frac{1}{sC}} = \frac{E(s) - V_{\text{out}}(s)}{R} \]

\[ sRC V_{\text{out}}(s) = E(s) - V_{\text{out}}(s) \]

\[ V_{\text{out}}(s)(sRC + 1) = E(s) \]

\[ H(s) = \frac{V_{\text{out}}(s)}{E(s)} = \frac{1}{sRC + 1} \]

s-plane transfer function

bilinear transform

\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = \frac{2}{T} \text{ (typically)} \]

z-plane transfer function

**DISCRETIZE RC NETWORK** (traditional approach)

\[ V_{\text{out}}(s) = \frac{E(s) - V_{\text{out}}(s)}{sRC} \]

\[ V_{\text{out}}(s) (sRC + 1) = E(s) \]

\[ H(s) = \frac{V_{\text{out}}(s)}{E(s)} = \frac{1}{sRC + 1} \]

**bilinear transform**

\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \quad \text{(typically)} \]

**z-plane transfer function**

\[ H(z) = \frac{V_{\text{out}}(z)}{E(z)} = \frac{T}{T+2RC} + \frac{T}{T+2RC} \frac{z^{-1}}{z^{-1}} \]

**Discretize RC Network** (traditional approach)

\[
H(z) = \frac{V_{\text{out}}(z)}{E(z)} = \frac{T}{T+2RC} + \frac{T}{T+2RC} z^{-1} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}
\]

with
\[
\begin{align*}
    b_0 &= \frac{T}{T+2RC} \\
    b_1 &= \frac{T}{T+2RC} \\
    a_1 &= \frac{T}{T-2RC} \\
\end{align*}
\]

---

**Direct Form I**

- Input: \(x(n)\)
- Coefficients: \(b_0\) and \(b_1\)
- Delay: \(z^{-1}\)

**Direct Form II**

- Input: \(x(n)\)
- Coefficients: \(b_0\) and \(b_1\)
- Delay: \(z^{-1}\)

**Transposed Direct Form I**

- Input: \(x(n)\)
- Coefficients: \(b_0\) and \(b_1\)
- Delay: \(z^{-1}\)

**Transposed Direct Form II**

- Input: \(x(n)\)
- Coefficients: \(b_0\) and \(b_1\)
- Delay: \(z^{-1}\)
Discretize RC Network (traditional approach)

\[
H(z) = \frac{V_{\text{out}}(z)}{E(z)} = \frac{T}{T+2RC} + \frac{T}{1 + \frac{T-2RC}{T+2RC}z^{-1}} = \frac{b_0 + b_1z^{-1}}{1 + a_1z^{-1}} \quad \text{with} \quad \begin{cases} 
b_0 = \frac{T}{T+2RC} \\
a_1 = \frac{T}{T+2RC} \\
b_1 = \frac{T}{T-2RC} \\
1 = \frac{T}{T+2RC} \end{cases}
\]

three multiplier form

two multiplier form

one multiplier form (A)

one multiplier form (B)

DISCRETIZE RC NETWORK (traditional approach)

- need transfer function
- factor into biquads if high order
- have to choose form for desired properties
DISCRETIZE RC NETWORK (traditional approach)

- need transfer function
- factor into biquads if high order
- have to choose form for desired properties
- issues with time-varying circuits

DISCRETIZE RC NETWORK (attempt modular)

what if...

• modular / topology-preserving?
• reusable?
• skip transfer function representation?

spoiler alert: this won’t work in the Kirchhoff domain...
DISCRETIZE RC NETWORK (attempt modular)

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• modular / topology-preserving?
• reusable?
• skip transfer function representation?

spoiler alert: this won’t work in the Kirchhoff domain...
DISCRETIZE RC NETWORK (attempt modular)
DISCRETIZE RC NETWORK (attempt modular)
**DISCRETIZE RC NETWORK (attempt modular)**

The diagram shows a simple RC circuit with an input voltage $e(t)$ and an output voltage $v_{out}(t)$. The underlying structure of the circuit is depicted, showing the resistive voltage source and the capacitor in parallel connection. The attempt to modularize the network is illustrated, indicating an attempt to simplify or break down the circuit into more manageable parts.
DISCRETIZE RC NETWORK (attempt modular)
DISCRETIZE RC NETWORK (attempt modular)
**Discretize RC Network** (attempt modular)

Diagram showing a resistive voltage source, parallel connection, and a capacitor.
**DISCRETIZE RC NETWORK (attempt modular)**

\[ v_1 = e(t) + R \cdot i_1 \]
**Discretize RC Network (attempt modular)**

**Resistive Voltage Source**

\[ v_1 = e(t) + R \cdot i_1 \]

**Parallel Connection**

**Capacitor**

\[ e(t) \rightarrow v_1 \]

\[ i_1 \rightarrow v_1 \]
**DISCRETIZE RC NETWORK** (attempt modular)

### Resistive Voltage Source

- **Equation:**
  \[ v_1 = e(t) + R \cdot i_1 \]

### Parallel Connection

- **Equations:**
  \[ v_3 = v_2 \]
  \[ i_2 = -i_3 \]

### Capacitor
DISCRETIZE RC NETWORK (attempt modular)

\[ v_1 = e(t) + R \cdot i_1 \]

\[ v_3 = v_2 \]

\[ i_2 = -i_3 \]
**Discretize RC Network (attempt modular)**

\[ e(t) \]

\[ v_1 = e(t) + R \cdot i_1 \]

\[ e(t) \rightarrow v_1 \]

\[ v_1 = e(t) + R \cdot i_1 \]

\[ i_1 \]

\[ v_2 \]

\[ v_3 = v_2 \]

\[ i_2 = -i_3 \]

\[ i_2 \rightarrow i_3 \]

\[ v_3 = v_2 \]

\[ i_2 \rightarrow i_3 \]

\[ i_4 = sC \cdot v_4 \]

\[ i_4 \]

\[ v_4 \]

\[ C \]

\[ v_3 \]

\[ v_2 \]
**Discretize RC Network (attempt modular)**

\[ v_1 = e(t) + R \cdot i_1 \]

\[ i_2 = -i_3 \]

\[ v_3 = v_2 \]

\[ i_4 = sC \cdot v_4 \]
DISCRETIZE RC NETWORK (attempt modular)

capacitor
(continuous time)

\[ v_4 \quad \downarrow \quad sC \quad \downarrow \quad i_4 \]
**DISCRETIZE RC NETWORK** (attempt modular)

- **Capacitor** (continuous time)

\[ V_4(s) = I_4(s) \frac{1}{sC} \]
**DISCRETIZE RC NETWORK** (attempt modular)

- **Capacitor** (continuous time)

\[ V_4(s) = I_4(s) \frac{1}{sC} \]

**Bilinear transform**

\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \text{ (typically)} \]

\[ V_4(z) = I_4(z) \frac{1}{2 \frac{1 - z^{-1}}{T} \frac{1}{1 + z^{-1}} C} \]
**Discretize RC Network** (attempt modular)

Capacitor (continuous time)

\[
V_4(s) = I_4(s) \frac{1}{sC}
\]

Bilinear transform \( s \leftarrow c \frac{1-z^{-1}}{1+z^{-1}}, \ c = 2/T \) (typically)

\[
V_4(z) = I_4(z) \frac{1}{2T} \frac{1}{1+z^{-1}} \frac{1}{C}
\]

\[
V_4(z) = I_4(z) \frac{T}{2C} \frac{1}{1-z^{-1}}
\]
**DISCRETIZE RC NETWORK** (attempt modular)

**capacitor**
(continuous time)

\[ V_4(s) = I_4(s) \frac{1}{sC} \]

**bilinear transform**
\[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \ c = \frac{2}{T} \text{ (typically)} \]

\[ V_4(z) = I_4(z) \frac{1}{\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} C} \]

\[ V_4(z) = I_4(z) \frac{T}{2C} \frac{1 + z^{-1}}{1 - z^{-1}} \]

**inverse z transform**
\[ x[n] = Z^{-1} \{ X(z) \} \]

\[ v_4[n] - v_4[n - 1] = \frac{T}{2C} i_4[n] + \frac{T}{2C} i_4[n - 1] \]
**Discretize RC Network** (attempt modular)

Capacitor (continuous time)

\[ V_4(s) = I_4(s) \frac{1}{sC} \]

**Bilinear transform**

\[ s \leftarrow c \frac{1-z^{-1}}{1+z^{-1}}, \quad c = \frac{2}{T} \text{ (typically)} \]

\[ V_4(z) = I_4(z) \frac{1}{2 \frac{1-z^{-1}}{1+z^{-1}} C} \]

\[ V_4(z) = I_4(z) \frac{T}{2C} \frac{1 + z^{-1}}{1 - z^{-1}} \]

**Inverse z transform**

\[ x[n] = Z^{-1} \{X(z)\} \]

**Difference equation**

\[ v_4[n] - v_4[n-1] = \frac{T}{2C} i_4[n] + \frac{T}{2C} i_4[n-1] \]

\[ i_4[n] = \frac{2C}{T} v_4[n] - \frac{2C}{T} v_4[n-1] - i_4[n-1] \]
**Discretize RC Network (attempt modular)**

**Capacitor (continuous time)**

![Capacitor](image)

\[ V_4(s) = I_4(s) \frac{1}{sC} \]

**Bilinear transform**

\[ s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = \frac{2}{T} \text{ (typically)} \]

\[ V_4(z) = I_4(z) \frac{1}{2C \frac{1 - z^{-1}}{1 + z^{-1}} C} \]

\[ V_4(z) = I_4(z) \frac{T}{2C} \frac{1 + z^{-1}}{1 - z^{-1}} \]

**Inverse z transform**

\[ x[n] = \mathcal{Z}^{-1} \{X(z)\} \]

\[ v_4[n] - v_4[n-1] = \frac{T}{2C} i_4[n] + \frac{T}{2C} i_4[n-1] \]

**Difference equation**

\[ i_4[n] = \frac{2C}{T} v_4[n] - \frac{2C}{T} v_4[n-1] - i_4[n-1] \]
**DISCRETIZE RC NETWORK** (attempt modular)

---

\[ e(t) \uparrow R \downarrow C \downarrow v_{out}(t) \]

---

\[ e(t) \uparrow R \downarrow i_1 \quad i_2 \quad i_3 \quad i_4 \downarrow C \downarrow v_{out}(t) \]
**DISCRETIZE RC NETWORK (attempt modular)**

![Diagrams of RC network and its discretized version]
**DISCRETIZE RC NETWORK (attempt modular)**

\[ e(t) \quad R \quad C \quad v_{out}(t) \]

\[ e(t) \quad R \quad i_1 \quad v_1 \quad v_2 \quad i_2 \quad v_3 \quad i_3 \quad v_4 \quad C \quad v_{out}(t) \]

discretized

**delay-free loop!**
structure is noncomputable
**Discretize RC Network** (**WDF approach**)

**WDF approach** involves:

- introduction of free parameter (port resistance) at each port:
  \[ R_n > 0, \text{ for each port } n \]

- introduction of wave variables:
  \[ a_n = v_n + R_n i_n \]
  \[ b_n = v_n - R_n i_n \]

- discretization of reactive elements (capacitors, inductors) using the Bilinear transformation:
  \[ s \leftarrow c \frac{1 - z^{-1}}{1 + z^{-1}}, \quad c = 2/T \text{ (typically)} \]
**DISCRETIZE RC NETWORK (WDF approach)**

port resistance:
incident wave:
reflected wave:
**DISCRETIZE RC NETWORK (WDF approach)**

- **Resistive voltage source:**
  - Port resistance: $R_1$
  - Incident wave: $a_1 = v_1 + R_1 i_1$
  - Reflected wave: $b_1 = v_1 - R_1 i_1$

- **Parallel connection:**

- **Capacitor:**
  - $C$

Diagram showing an RC network with resistive sources and capacitors connected in parallel.
**Discretize RC Network (WDF approach)**

- **Resistive Voltage Source**
  -\[ e(t) \]
  -\[ R \]
  -\[ i_1 \]
  -\[ v_1 \]

- **Parallel Connection**
  -\[ i_2 \]
  -\[ i_3 \]
  -\[ v_2 \]
  -\[ v_3 \]

- **Capacitor**

**Port Resistance:**
- \[ R_1 \]

**Incident Wave:**
- \[ a_1 = v_1 + R_1 i_1 \]
- \[ b_1 = v_1 - R_1 i_1 \]

**Reflected Wave:**
- \[ a_2 = v_2 + R_2 i_2 \]
- \[ b_2 = v_2 - R_2 i_2 \]
**Discretize RC Network (WDF approach)**

- **Resistive voltage source**
  - Port resistance: $R_1$
  - Incident wave: $a_1 = v_1 + R_1 i_1$
  - Reflected wave: $b_1 = v_1 - R_1 i_1$

- **Parallel connection**
  - Port resistance: $R_2$
  - Incident wave: $a_2 = v_2 + R_2 i_2$
  - Reflected wave: $b_2 = v_2 - R_2 i_2$

- **Capacitor**
  - Port resistance: $R_3$
  - Incident wave: $a_3 = v_3 + R_3 i_3$
  - Reflected wave: $b_3 = v_3 - R_3 i_3$
**DISCRETIZE RC NETWORK (WDF approach)**

resistive voltage source

parallel connection

port resistance:

incident wave:

reflected wave:

<table>
<thead>
<tr>
<th>Port</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$a_1 = v_1 + R_1 i_1$</td>
</tr>
<tr>
<td></td>
<td>$b_1 = v_1 - R_1 i_1$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$a_2 = v_2 + R_2 i_2$</td>
</tr>
<tr>
<td></td>
<td>$b_2 = v_2 - R_2 i_2$</td>
</tr>
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<td>$R_3$</td>
<td>$a_3 = v_3 + R_3 i_3$</td>
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<td>$a_4 = v_4 + R_4 i_4$</td>
</tr>
<tr>
<td></td>
<td>$b_4 = v_4 - R_4 i_4$</td>
</tr>
</tbody>
</table>
**Discretize RC Network** (WDF approach)
DISCRETIZE RC NETWORK (WDF approach)

resistive
voltage
source

\[ e(t) \]

\[ R \]

\[ i_1 \]

\[ v_1 \]
**DISCRETIZE RC NETWORK (WDF approach)**

\[ v_1 = e(t) + R i_1 \]
**Discretize RC Network (WDF approach)**

Resistive voltage source

\[ v_1 = e(t) + Ri_1 \]

\[ \frac{a_1}{2} + \frac{b_1}{2} = e(t) + R \frac{a_1}{2R_1} - R \frac{b_1}{2R_1} \]
**DISCRETIZE RC NETWORK (WDF approach)**

- **Resistive voltage source**

\[
\begin{align*}
\text{v}_1 &= e(t) + R\text{i}_1 \\
\frac{a_1}{2} + \frac{b_1}{2} &= e(t) + R \frac{a_1}{2R_1} - R \frac{b_1}{2R_1} \\
b_1 (R + R_1) &= 2R_1 e(t) + a_1 (R - R_1)
\end{align*}
\]

**Difference equation**

\[
b_1 = \frac{2R_1}{R + R_1} e(t) + \frac{R - R_1}{R + R_1} a_1
\]
**Discretize RC Network (WDF approach)**

Resistive voltage source:

\[ v_1 = e(t) + Ri_1 \]

\[
\frac{a_1}{2} + \frac{b_1}{2} = e(t) + R \frac{a_1}{2R_1} - R \frac{b_1}{2R_1}
\]

\[ b_1(R + R_1) = 2R_1e(t) + a_1(R - R_1) \]

Difference equation:

\[ b_1 = \frac{2R_1}{R + R_1}e(t) + \frac{R - R_1}{R + R_1}a_1 \]
DISCRETIZE RC NETWORK (WDF approach)

parallel connection

\[ i_2 \quad i_3 \]

\[ v_2 \quad v_3 \]
**DISCRETIZE RC NETWORK** (WDF approach)

![Parallel connection diagram]

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]
**DISCRETIZE RC NETWORK (WDF approach)**

\[
\text{parallel connection}
\]

\[
\begin{align*}
i_2 &= -i_3 \\
v_2 &= v_3 \\
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 &= -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3 \\
\frac{1}{2} a_2 + \frac{1}{2} b_2 &= \frac{1}{2} a_3 + \frac{1}{2} b_3
\end{align*}
\]
**Discretize RC Network (WDF approach)**

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]

\[
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 = -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3 \\
\frac{1}{2} a_2 + \frac{1}{2} b_2 = \frac{1}{2} a_3 + \frac{1}{2} b_3
\]

\[ R_3 a_2 - R_3 b_2 = -R_2 a_3 + R_2 b_3 \]
\[ a_2 + b_2 = a_3 + b_3 \]
**DISCRETIZE RC NETWORK (WDF approach)**

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]

\[
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 = -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3
\]
\[
\frac{1}{2} a_2 + \frac{1}{2} b_2 = \frac{1}{2} a_3 + \frac{1}{2} b_3
\]

\[ R_3 a_2 - R_3 b_2 = -R_2 a_3 + R_2 b_3 \]
\[ a_2 + b_2 = a_3 + b_3 \]

\[
\begin{bmatrix} R_3 & R_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} R_3 & R_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
\]
**Discretize RC Network (WDF approach)**

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]

\[
\begin{align*}
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 &= -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3 \\
\frac{1}{2} a_2 + \frac{1}{2} b_2 &= \frac{1}{2} a_3 + \frac{1}{2} b_3 \\
R_3 a_2 - R_3 b_2 &= -R_2 a_3 + R_2 b_3 \\
a_2 + b_2 &= a_3 + b_3
\end{align*}
\]

\[
\begin{bmatrix}
R_3 & R_2 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
b_2 \\
b_3
\end{bmatrix}
= 
\begin{bmatrix}
R_3 & R_2 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_2 \\
b_3
\end{bmatrix}
= 
\frac{1}{R_2 + R_3}
\begin{bmatrix}
1 & -R_2 \\
1 & R_3
\end{bmatrix}
\begin{bmatrix}
R_3 & R_2 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
a_2 \\
a_3
\end{bmatrix}
\]
**Discretize RC Network (WDF approach)**

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]

\[
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 = -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3
\]
\[
\frac{1}{2} a_2 + \frac{1}{2} b_2 = \frac{1}{2} a_3 + \frac{1}{2} b_3
\]

\[ R_3a_2 - R_3b_2 = -R_2a_3 + R_2b_3 \]
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\begin{bmatrix} R_3 & R_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} R_3 & R_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
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\begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{1}{R_2 + R_3} \begin{bmatrix} 1 & -R_2 \\ 1 & R_3 \end{bmatrix} \begin{bmatrix} R_3 & R_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
\]

**scattering equation**
**Discretize RC Network (WDF approach)**

\[ i_2 = -i_3 \]
\[ v_2 = v_3 \]

\[
\frac{1}{2R_2} a_2 - \frac{1}{2R_2} b_2 = -\frac{1}{2R_3} a_3 + \frac{1}{2R_3} b_3
\]
\[
\frac{1}{2} a_2 + \frac{1}{2} b_2 = \frac{1}{2} a_3 + \frac{1}{2} b_3
\]

\[
R_3 a_2 - R_3 b_2 = -R_2 a_3 + R_2 b_3
\]
\[
a_2 + b_2 = a_3 + b_3
\]

\[
\begin{bmatrix} R_3 & R_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} R_3 & R_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
\]

\[
\begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{1}{R_2 + R_3} \begin{bmatrix} 1 & -R_2 \\ 1 & R_3 \end{bmatrix} \begin{bmatrix} R_3 & R_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
\]

**Scattering equation**

\[
\begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \frac{R_2 - R_3}{R_2 + R_3} & \frac{2R_2}{R_2 + R_3} \\ -\frac{R_2 - R_3}{R_2 + R_3} & \frac{2R_2}{R_2 + R_3} \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}
\]
DISCRETIZE RC NETWORK (WDF approach)

\[ V_4(s) = \frac{1}{sC} I_4(s) \]
**Discretize RC Network (WDF approach)**

\[ V_4(s) = \frac{1}{sC} I_4(s) \]

\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]
**Discretize RC Network (WDF approach)**

\[ V_4(s) = \frac{1}{sC} I_4(s) \]

\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]

\[ \left( \frac{1}{2} A_4(z) + \frac{1}{2} B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( \frac{1}{2R_4} A_4(z) - \frac{1}{2R_4} B_4(z) \right) (1 + z^{-1}) \]
DISCRETIZE RC NETWORK (WDF approach)

\[ V_4(s) = \frac{1}{sC} I_4(s) \]
\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]
\[
\left( \frac{1}{2} A_4(z) + \frac{1}{2} B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( \frac{1}{2R_4} A_4(z) - \frac{1}{2R_4} B_4(z) \right) (1 + z^{-1})
\]
\[
R_4 \left( A_4(z) + B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( A_4(z) - B_4(z) \right) (1 + z^{-1})
\]
**Discretize RC Network (WDF approach)**

\[ V_4(s) = \frac{1}{sC} I_4(s) \]

\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]

\[ \left( \frac{1}{2} A_4(z) + \frac{1}{2} B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( \frac{1}{2R_4} A_4(z) - \frac{1}{2R_4} B_4(z) \right) (1 + z^{-1}) \]

\[ R_4 \left( A_4(z) + B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( A_4(z) - B_4(z) \right) (1 + z^{-1}) \]

\[ \left( R_4 + \frac{T}{2C} \right) B_4(z) = \left( R_4 - \frac{T}{2C} \right) B_4(z) z^{-1} + \left( -R_4 + \frac{T}{2C} \right) A_4(z) + \left( R_4 + \frac{T}{2C} \right) A_4(z) z^{-1} \]
**Discretize RC Network (WDF approach)**

\[ V_4(s) = \frac{1}{sC} I_4(s) \]

\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]

\[
\left( \frac{1}{2} A_4(z) + \frac{1}{2} B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( \frac{1}{2R_4} A_4(z) - \frac{1}{2R_4} B_4(z) \right) (1 + z^{-1})
\]

\[
R_4 (A_4(z) + B_4(z)) (1 - z^{-1}) = \frac{T}{2C} (A_4(z) - B_4(z)) (1 + z^{-1})
\]

\[
\left( R_4 + \frac{T}{2C} \right) B_4(z) = \left( R_4 - \frac{T}{2C} \right) B_4(z) z^{-1} + \left( -R_4 + \frac{T}{2C} \right) A_4(z) + \left( R_4 + \frac{T}{2C} \right) A_4(z) z^{-1}
\]

**Difference equation**

\[
b_4[n] = \frac{R_4 - \frac{T}{2C}}{R_4 + \frac{T}{2C}} b[n - 1] - \frac{R_4 - \frac{T}{2C}}{R_4 + \frac{T}{2C}} a_4[n] + a_4[n - 1]
\]
**Discretize RC Network (WDF approach)**

\[ V_4(s) = \frac{1}{sC} I_4(s) \]

\[ V_4(z)(1 - z^{-1}) = \frac{T}{2C} I_4(z)(1 + z^{-1}) \]

\[ \left( \frac{1}{2} A_4(z) + \frac{1}{2} B_4(z) \right) (1 - z^{-1}) = \frac{T}{2C} \left( \frac{1}{2R_4} A_4(z) - \frac{1}{2R_4} B_4(z) \right) (1 + z^{-1}) \]

\[ R_4 (A_4(z) + B_4(z)) (1 - z^{-1}) = \frac{T}{2C} (A_4(z) - B_4(z)) (1 + z^{-1}) \]

\[ \left( R_4 + \frac{T}{2C} \right) B_4(z) = \left( R_4 - \frac{T}{2C} \right) B_4(z) z^{-1} + \left( -R_4 + \frac{T}{2C} \right) A_4(z) + \left( R_4 - \frac{T}{2C} \right) A_4(z) z^{-1} \]

**Difference equation**

\[ b_4[n] = \frac{R_4 - \frac{T}{2C}}{R_4 + \frac{T}{2C}} b[n - 1] - \frac{R_4 - \frac{T}{2C}}{R_4 + \frac{T}{2C}} a_4[n] + a_4[n - 1] \]
DISCRETIZE RC NETWORK (WDF approach)
DISCRETIZE RC NETWORK (WDF approach)

delay-free loops!
structure is noncomputable
DISCRETIZE RC NETWORK (WDF approach)

delay-free loops!
structure is noncomputable

but, this time, we can fix things!
by tuning $R_1$, $R_2$, $R_3$, $R_4$
but, this time, we can fix things!
by tuning $R$
**DISCRETIZE RC NETWORK (WDF approach)**

\[ R_3 = R_4 \]  
(by def.)

\[ R_4 = \frac{T}{2C} \]

**but, this time, we can fix things!**

by tuning \( R \)
**Discretize RC Network (WDF approach)**

\[ R_2 = \frac{T}{2C} \]

\[ R_3 = R_4 \quad \text{(by def.)} \]

\[ R_4 = \frac{T}{2C} \]

*but, this time, we can fix things!*

*by tuning R*
**DISCRETIZE RC NETWORK (WDF approach)**

\[ R_1 = R_2 \quad \text{(by def.)} \]

\[ R_2 = \frac{T}{2C} \]

\[ R_3 = R_4 \quad \text{(by def.)} \]

\[ R_4 = \frac{T}{2C} \]

**structure is computable!**
WAVE DIGITAL FILTERS

- modular
- no transfer function representation needed
- no factoring into biquads needed

- structure arranged as a “tree”
  - one element as the root, resolve loops upwards

- energetic properties in reference domain used to guarantee stability by construction
  \[ p_n = \frac{a_n^2 - b_n^2}{R_n} \]
- good on quantization/sensitivity (original purpose)

Wave Digital Filters (resolved one ports)

(a) A generic one-port
(b) resistor: \( R_p = R \)
(c) capacitor: \( R_p = T/2C \)
(d) inductor: \( R_p = 2L/T \)
(e) voltage source \( R_p = R \)

\( C, L, R, \) are physical capacitance, inductance, and resistance, \( T \) is unit delay

Wave Digital Filters (adaptors)

Parallel Adaptor

\[ \gamma_v = \frac{2G_v}{G_1 + G_2 + G_3}, \quad v = 1, 2 \]

Series Adaptor

\[ \gamma_v = \frac{2R_v}{R_1 + R_2 + R_3}, \quad v = 1, 2 \]

**WAVE DIGITAL FILTERS (binary connection tree)**

- binary connection tree (BCT) systematizes WDF with only series and parallel connections
- up to one nonlinearity
- N-port series connections implemented with \( (N-2) \) 3-port series adaptors
- N-port parallel connections implemented with \( (N-2) \) 3-port parallel adaptors

Fig. 5. Two examples of BCTs. (left) Generic one and (right) chainlike circuit. The circular box represents an instantaneous adaptor, in which the adapted port is clearly specified. This particular notational choice simplifies the drawing of connection trees with a great amount of branching.

INTRODUCTION

1. tutorial review of WDF principles
2. recent theoretical progress in WDFs
3. WDF software overview and demo
CURRENT RESEARCH at CCRMA

- **Wave Digital Filter Adaptors for Arbitrary Topologies and Multiport Linear Elements**
  @ Int. Conf. Digital Audio Effects (DAFx-15), Trondheim, Norway, Nov. 30 – Dec. 3, 2015
  ↳ Kurt James Werner, Julius O. Smith III, and Jonathan Abel

- **Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities**
  @ Int. Conf. Digital Audio Effects (DAFx-15), Trondheim, Norway, Nov. 30 – Dec. 3, 2015
  ↳ Kurt James Werner, Vaibhav Nangia, Julius O. Smith III, and Jonathan Abel

- **A General and Explicit Formulation for Wave Digital Filters with Multiple/Multiport Nonlinearities and Complicated Topologies**
  ↳ Kurt James Werner, Vaibhav Nangia, Julius O. Smith III, and Jonathan Abel

- **An Improved and Generalized Diode Clipper Model for Wave Digital Filters**
  @ AES 139th Convention, New York, USA, Oct. 29 – Nov. 1, 2015
  ↳ Kurt James Werner, Vaibhav Nangia, Alberto Bernardini, Julius O. Smith III, and Augusto Sarti

- **An Energetic Interpretation of Nonlinear Wave Digital Filter Lookup Table Error**
  @ IEEE Int. Symp. Signals, Circuits, Syst. (ISSCS), Iasi, Romania, July 9–10, 2015
  ↳ Kurt James Werner and Julius O. Smith III
CURRENT RESEARCH at Politecnico di Milano

- Modeling Nonlinear Wave Digital Elements using the Lambert Function
  (*submitted* to IEEE Transactions on Circuits and Systems I: Regular Papers)
  ↪ Alberto Bernardini, Kurt James Werner, Augusto Sarti, and Julius O. Smith III

- Modeling a Class of Multi-Port Nonlinearities in Wave Digital Structures
  @ European Signal Process. Conf. (EUSIPCO), Nice, France, August 31, 2015
  ↪ Alberto Bernardini, Kurt James Werner, Augusto Sarti, and Julius O. Smith III

- Multi-Port Nonlinearities in Wave Digital Structures
  @ IEEE Int Symp. Signals, Circuits, Syst. (ISSCS), Iasi, Romania, July 9–10, 2015
  ↪ Alberto Bernardini, Kurt James Werner, Augusto Sarti, and Julius O. Smith III

- Modeling Nonlinear Circuits with Multi-port Elements in the Wave Digital Domain
  Master’s thesis, Politecnico di Milano, Italy, April 2015
  ↪ Alberto Bernardini
NONLINEARITIES IN WDFs

1. single nonlinearity
2. consolidated one-port combination
3. cross-controlled multiport
4. simplified multiports
5. linearized multiport
6. piecewise linear models
7. iterative schemes
NONLINEARITIES IN WDFs: single nonlinearity

- accommodate **one** one-port NL element, e.g.:
  - ideal rectifier (ideal diode)
  - piecewise linear resistance
- can view as lookup table with interpolation or piecewise linear segments
- must solve $b = f(a)$ at root

NONLINEARITIES IN WDFs: single nonlinearity

Kirchhoff ($i$–$v$) domain

Voltage wave transformation

wave ($a$–$b$) domain

Fig. 3. Characteristic of the nonlinear resistance defined by (16).

$$i = G_1 v + \frac{1}{2} (G_2 - G_1) (|v + v_0| - |v - v_0|), \quad (16)$$

Fig. 5. Plot of the characteristic defined by (17).

$$b = e(a) = e_1 a + \frac{1}{2} (e_2 - e_1) (|a + a_0| - |a - a_0|), \quad (17)$$

NONLINEARITIES IN WDFs: single nonlinearity

- use “mutators” from classical network theory to enable, e.g., nonlinear $q$–$v$ relationships
- these are needed for nonlinear elements “with memory”
- for example, nonlinear capacitors and inductors where flux or charge can saturate
NONLINEARITIES IN WDFs: consolidated one-ports

- multiple nonlinearities handled by consolidating into a single one-port
- implicit nonlinear function solved as $b = f(a)$ with numerical methods

\[
\frac{b - a}{2R} = I_s \left( e^{\frac{a+b}{2V_T}} - 1 \right) - I_s \left( e^{-\frac{a+b}{2V_T}} - 1 \right)
\]

Figure 6: Schematic of the diode clipper with high-pass and low-pass capacitors.

Figure 7: WDF tree of the two-capacitor diode clipper. Diode D is root.

NONLINEARITIES IN WDFs: consolidated one-ports

\[ b = f(a) \] for diode pair solved using Lambert \( W \) function, assuming one diode dominates: (ignoring saturation current)

- One diode (Kirchhoff): 
  \[ i = I_s \left( e^{\frac{v}{V_T}} - 1 \right) \]
- One diode (wave, implicit): 
  \[ \frac{a - b}{2R} = I_s \left( e^{\frac{a+b}{2V_T}} - e^{-\frac{a+b}{2V_T}} \right) \]
- One diode (wave, explicit): 
  \[ b = f(a) = a + 2RI_s - 2V_T W \left( \frac{RI_s}{V_T} e^{\frac{RI_s+a}{V_T}} \right) \]
- Diode pair (approximate): 
  \[ b = \text{sgn}(a) \cdot f(|a|) \]

**NONLINEARITIES IN WDFs: consolidated one-ports**

- explicit model improved by canceling some approximation error of Paiva et al. (2013) model with an additional Lambert $W$ term
- generalized to any number of diodes in each direction (stock and hacked guitar distortion pedals)

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**NONLINEARITIES IN WDFs**: consolidated one-ports

Linear secant interpolation and tangent extrapolation can be incrementally (gray) or globally (thatched) non-passive.

Choosing table points and secant/tangent properly (considering \( \text{sgn}(a) \) and \( a'' \)) yields interpolation methods that respect passivity.

NONLINEARITIES IN WDFs: cross-controls

- grid–cathode voltage modeled as “cross-control” of plate–cathode nonlinearity, with ad-hoc delay to aid realizability

Nonlinearities in WDFs: cross-controls

- frame BJT as two-port nonlinear element
- two linear WDF subtrees
- Emitter voltage $V_E$ proposed as “cross-control” on BJT
NONLINEARITIES IN WDFs : linearized multiport

- proposes using linearized Hybrid-π model of BJT
- three linear WDF subtrees

NONLINEARITIES IN WDFs: piecewise linear models

- represent vector of nonlinear root elements with piecewise linear approximation
- limited to vector parallel relationship between “internal” (a and b) and “external” (a_1 and b_1) root ports

**Nonlinearities in WDFs**: piecewise linear models

- addresses Petrausch & Rabenstein (2004) limit to vector parallel case for multiple one-port nonlinearities

NONLINEARITIES IN WDFs: simplified multiports

- depending on operating point, transport across particular $p$–$n$ junctions in a BJT can be reasonably neglected
- introduces some approximation error, but renders equations tractable using the Lambert $W$, as in Paiva et al. (2013)
- Opportunities for treating cases with feedback


**Nonlinearities in WDFs: iterative schemes**

- entire triode nonlinearity contained in root element
- three linear WDF subtrees (1 2 3)
- root solved with customized secant method (specific to triode model)

Nonlinearities in WDFs: iterative schemes

- Multiple nonlinearities create delay-free loops
- Resolved by inserting extra delay elements as second time dimensions ($T_2$)
- Framed as extension to multidimensional case
- $T_2$'s solved by iteration, convergence guaranteed by contractivity of WDF properties energy metric

NONLINEARITIES IN WDFs: iterative schemes

- same technique applied to topological problems in linear circuits (e.g., bridged-T topology)

NONLINEARITIES IN WDFs

- recent research focuses on multiple nonlinearities
- since 1989, we’ve known that >1 requires especial treatment
- solutions with a single multiport at the root seem promising
- what about MULTIPLE multiports?
- ad hoc solutions sometimes work, but no general solutions
**TOPOLOGICAL ASPECTS OF WDFs**

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Figure 2: Deriving a WDF adaptor structure for the Big Muff Pi clipping stage.
**Topological Aspects of WDFs**

WDFs originally intended for *ladder* and *lattice* circuits (topology comprised of only series and parallel)

in general, circuits have ANY topology *(infinite class of other types of connections)*
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**TOPOLOGICAL ISSUES IN WDFs:** SPQR tree

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PROBLEMS with Wave Digital Filters

1. No general method for deriving topology
2. No general method for handling complicated topologies
3. No general method for handling multiple nonlinearities

• Wave Digital Filter Adaptors for Arbitrary Topologies and Multiport Linear Elements
  @ Int. Conf. Digital Audio Effects (DAFx-15), Trondheim, Norway, Nov. 30 – Dec. 3, 2015
  ↦ Kurt James Werner, Julius O. Smith III, and Jonathan Abel

• Resolving Wave Digital Filters with Multiple/Multiport Nonlinearities
  @ Int. Conf. Digital Audio Effects (DAFx-15), Trondheim, Norway, Nov. 30 – Dec. 3, 2015
  ↦ Kurt James Werner, Vaibhav Nangia, Julius O. Smith III, and Jonathan Abel

• A General and Explicit Formulation for Wave Digital Filters with Multiple/Multiport Nonlinearities and Complicated Topologies
  ↦ Kurt James Werner, Vaibhav Nangia, Julius O. Smith III, and Jonathan Abel
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PROBLEMS WITH WAVE DIGITAL FILTERS

1. No general method for deriving topology
2. No general method for handling complicated topologies
3. No general method for handling multiple nonlineairities


INTRODUCTION

1. Tutorial review of WDF principles
2. Recent theoretical progress in WDFs
3. WDF software overview and demo
SUMMARY

• WDFs are an elegant solution for circuit modeling

• Frustratingly applicable to only a tiny class of circuits

• Outside that class, ad hoc solutions (focused on nonlinearities) dominate

• New research addressing topological issues (details at talks tomorrow!) vastly expands the range of suitable reference circuits
Thank you for listening!

...now go build some WDFs!

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