Overburden dependent AVA inversion

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ABSTRACT
Amplitude-variation-with-offset (AVO) analysis is strongly dependent on interpretation of the estimated traveltime parameters. In practice, we can estimate two or three traveltime parameters that require interpretation within the families of two- or three-parameter velocity models, respectively. Increasing the number of model parameters improves the quality of overburden description and reduces errors in AVO analysis. We have analyzed the effect of two- and three-parameter velocity model interpretation for the overburden on AVO data and have developed error estimates in the reservoir parameters.

INTRODUCTION
Variations of seismic reflection coefficients with offset (amplitude variation with offset, or AVO) or incident angle (amplitude variation with angle, or AWA) play an important role in seismic interpretation as gas or hydrocarbon indicators (Ostrander, 1984; Swan, 1993). AVO/AVA attributes obtained by two- (Shuey, 1985) or three-term (Aki and Richards, 1980) AVO/AVA inversion are widely used in industry.

However, quality and accuracy of the inversion are affected by various factors. For example, Mora and Biondi (2000) investigate the sensitivity of AVO attributes to uncertainty in migration velocity. They consider various effects, including modeling, overburden, migration, velocity anomalies, and velocity errors. Xu et al. (1993) show that an inhomogeneous overburden interpreted as homogeneous in velocity analysis causes significant errors (up to 13%) in AVA inversion results.

Conventional velocity analysis assumes that all moveouts are hyperbolic and therefore only two traveltime parameters (zero-offset two-way traveltime and normal-moveout velocity) can be estimated during NMO correction. Applying the Dix (1955) equation results in two model parameters (thickness and velocity of the layer) that describe a constant-velocity layer.

However, in real media, the velocity distribution in the overburden is more complex. Hyperbolic velocity analysis results in wrong velocity reconstruction in the overburden that leads to incorrect raypath trajectory and offset-to-angle conversion, the source of error in AVO inversion. Moreover, errors in the raypath trajectory cause errors in the geometrical spreading correction on the amplitude data that additionally affect inversion results.

Nonhyperbolic velocity analysis that uses additional traveltime parameters called heterogeneity coefficients (Fomel and Grechka, 2001) improves velocity profile description. Practically, we can estimate only one additional traveltime parameter because of the quality of the seismic data and limited offset spread. Therefore, velocity reconstruction is limited to the family of three-parameter models (Stovas, 2008, 2009).

In this article, we investigate effects on AVO data caused by hyperbolic and nonhyperbolic velocity analysis. For simplicity, we consider a two-layer model whose upper layer has a linear gradient in the P-wave velocity and whose lower layer is a constant-velocity reservoir. We compare two- and three-parameter power-gradient velocity-model (Stovas, 2009) interpretations for this model and verify which interpretation gives more accurate AVO inversion results. For illustration, we provide the results of AVO inversion and AWA attribute estimation.

THEORY
Consider a vertically heterogeneous velocity model for the overburden, with the target interface represented by a horizontal reflector. From this interface, we record the reflection with travelt ime t(x) and amplitude function R(x). To perform the AVO inversion from these data, we first need to compute the travelt ime parameters from mov eout t(x) and then invert them for the model parameters.

The expansion of the reflection moveout squared in the Taylor series with respect to offset can be given in terms of heterogeneity coefficients (Taner and Koehler, 1969; Fomel and Grechka, 2001):
KINEMATICALLY EQUIVALENT MODELS

Stovas (2008) introduces the family of the kinematically equivalent velocity distributions that have a limited number of equal traveltime parameters.

To invert traveltime parameters within the framework of the three-parameter model, we are free to choose any model from the family of three-parameter kinematically equivalent velocity distributions that have the same traveltime parameters $t_0, V_{NMO}$, and $S_2$. Ignoring $S_2$ reduces to the two-parameter family of kinematically equivalent models.

The three-parameter family of velocity models is sufficient to account for all possible models ($-\infty < n < \infty$). The parameter $n$ describes the curvature of the velocity function $V(z)$. So, for $n = 1$, the curvature of the velocity function is zero. In general, the sign of the curvature is defined by sign $(1 - n)$. The sign of the curvature in the velocity function indicates behavior of the sedimentation that can be described by the rate of sedimentation, porosity, size of grains, etc. Thus, zero curvature implies uniform changes in sedimentation behavior, positive curvature indicates small changes in sedimentation parameters at large depths with large changes at shallow depths, and negative curvature corresponds to large changes in sedimentation behavior at large depths with small changes at shallow depths. Therefore, any geologic information regarding sedimentation behavior can be useful for choosing a model.

We consider five analytical kinematically equivalent models of the three-parameter family by setting $n = -2, -1, 1, 2$ in equation 2, where $n = -2, 0, 1$ corresponds to well-known linear sloth velocity, exponential velocity, and linear velocity models, respectively. We also consider the CV model by keeping only two traveltime parameters (this model can be obtained from equation 2 by taking the limits $n \rightarrow \pm \infty$). The two- and three-parameter kinematically equivalent models have different equations for traveltime parameter inversion (Appendix A). Therefore, the same traveltime parameters $t_0, V_{NMO}$, and $S_2$ result in different model parameters $H, V_0,$ and $\gamma$, depending on the value of $n$

The three-parameter kinematically equivalent models have different values for heterogeneity coefficients with orders larger than two. This can be illustrated by the expansion of higher-order heterogeneity coefficients in terms of the second-order heterogeneity coefficient (Appendix A):

$$S_k = 1 + \frac{k(k - 1)}{2}(S_2 - 1) + \frac{k(k - 1)(k - 2)(3 + 3k - 4n)}{40}(S_2 - 1)^2 + \cdots,$$

where $k = 3, 4, \ldots$. The first-order coefficient in series 3 does not depend on $n$, but the higher-order coefficients do; thus, they are different for different models from the family of the three-parameter equivalent velocity models. Decreasing the ratio $\gamma$ between the velocities at the bottom and top of the layer decreases all heterogeneity coefficients and lessens the effect of $n$ on the higher-order heterogeneity coefficients $S_k, k = 3, 4, \ldots$. 

The layer thickness in the series with respect to the second-order heterogeneity coefficient is given by (combining series A-8 and A-9):
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\[
H = \frac{V_{\text{NMO}}}{2} \left[ 1 - \frac{1}{8} (S_2 - 1) + \frac{3(9 - 8n)}{640} (S_2 - 1)^2 + \cdots \right].
\]

One can see that the main contribution comes from the first-order term. For vertically heterogeneous velocity models, \( S_2 \geq 1 \). So it is easy to see from equation 4 that layer thickness is less than in the case of a constant-velocity model (standard Dix equation). This follows from the inequality derived by Stovas (2009), \( \gamma^{-1} \leq 2H/(V_{\text{NMO}}) \leq 1 \).

RAY TRACING

To transform the data from AVO to A\( V^2 \) requires an offset-to-angle conversion. In a CV model, this equation is simple because of the straight rays in the medium. In three-parameter media, the seismic rays have different curved trajectories that result in different incidence angles at zero-depth and target-depth levels. At target level \( z = H \), the incident angle is defined by

\[
\sin(\theta_i) = pV_H = \gamma \sin(\theta_0),
\]

where \( \theta_0 \) and \( V_H \) are angle and velocity at the target level, respectively; \( \theta_0 \) is the incidence angle at the top of the layer.

Consider the expansion of the relation between the sine of the incident angle at target level \( \sin(\theta_H) \) and the normalized offset. From Appendix B, we obtain

\[
\sin(\theta_H) = \gamma \lambda \left[ \frac{1}{2} S_2 \bar{x}^3 + \frac{3}{4} \left( \frac{S_2 - 1}{2} \right) \bar{x}^3 + \cdots \right],
\]

where \( \lambda = V_0/V_{\text{NMO}} \), which implies the inequality \( \gamma^{-1} \leq \lambda \leq 1 \) (Stovas, 2009). The parameter \( \lambda \) can be expanded into the series with respect to the second-order heterogeneity coefficient (Appendix A):

\[
\lambda = 1 - \sqrt{3} (S_2 - 1)^{1/2} + \frac{3 - 2n}{8} (S_2 - 1)

+ \frac{n(1 - 3n)}{80} \sqrt{3}(S_2 - 1)^{3/2} + \cdots.
\]

The velocity ratio \( \gamma \) can also be expressed as a series with respect to the second-order heterogeneity coefficient (Appendix A):

\[
\gamma = 1 + \sqrt{3}(S_2 - 1)^{1/2} + \frac{3}{2} (S_2 - 1)

- \frac{n(5 + 3n + n^2)}{40} \sqrt{3}(S_2 - 1)^{3/2} + \cdots.
\]

The series for the product \( \gamma \lambda \) results in

\[
\gamma \lambda = 1 + \sqrt{3}(S_2 - 1)^{1/2} + \frac{3 - 2n}{8} (S_2 - 1)

- \frac{n(1 - 3n)}{80} \sqrt{3}(S_2 - 1)^{3/2} + \cdots.
\]

Among all heterogeneity coefficients in series 6, only \( S_2 \) is the same for all models because of kinematic equivalence, whereas \( S_k, k = 3, 4, \ldots \), are model dependent. This results in variable offset-to-angle relationships that impose an offset-dependent stretching factor on the A\( V^2 \) data.

Series 6 shows that the discrepancy between kinematically equivalent models results in different stretching factors and amplifies with increasing offset and velocity ratio \( \gamma \).

GEOMETRICAL SPREADING

The velocity models defined in equation 2 have ray trajectories that depend on the parameter \( n \). This affects the geometrical spreading factor. Assuming that the source and receiver are placed at the same depth, the relative geometrical spreading is (Ursin, 1990; Stovas and Ursin, 2009)

\[
L(x) = \cos \theta_0 \left[ \frac{1}{x} \frac{dt}{dx} \right]^{-1/2} = \cos \theta_0 \left[ \frac{x}{p} \frac{dx}{dp} \right]^{1/2},
\]

where \( \theta_0 \) is the incidence angle at the top of the layer. The first and second terms in equation 10 can be expanded in a series with respect to the normalized offset (Appendix B):

\[
\cos^2(\theta_0) = 1 - \lambda^2 \left[ \frac{\bar{x}^3}{3} - S_2 \bar{x}^4 + \frac{7S_2^2 - 3S_3}{4} \bar{x}^5 + \cdots \right],
\]

\[
\left[ \frac{x}{p} \frac{dx}{dp} \right]^{1/2} = V_{\text{NMO}}^2 \left[ 1 + S_2 \bar{x}^3 + \frac{9}{8} (S_1 - S_2^2) \bar{x}^4 + \cdots \right].
\]

Note that series 11 are valid for an arbitrary vertically heterogeneous medium. The series show that geometrical spreading depends on the higher-order heterogeneity coefficients and \( \lambda \). The three-parameter kinematically equivalent models have different \( \lambda \) and \( S_k, k = 3, 4, \ldots \). The discrepancy in geometrical spreading among these models is more pronounced for large offset. Being applied in true-amplitude A\( V^2 \)-oriented processing, the geometrical spreading factor imposes offset-dependent scaling on the data.

NUMERICAL EXAMPLES

To illustrate the theory, we consider a two-layer model whose overburden has the P-wave velocity distribution given in equation 2 for \( n = 1 \) (linear velocity model) and whose reservoir is a constant-velocity layer with properties \( V_p = 2850 \text{ m/s}, V_S = 1600 \text{ m/s}, \) and \( p = 2100 \text{ kg/m}^3 \). Parameters of the overburden are \( V_0 = V_p(0) = 1800 \text{ m/s}, \gamma = 1.5, H = 1000 \text{ m}, V_p(H) = 1380 \text{ m/s}, \) and \( p_1 = 1800 \text{ kg/m}^3 \). We consider PP reflections only. The distribution of S-wave velocity and density in the overburden can be arbitrary because they do not affect P-wave propagation — only their contrast at the target level is important for A\( V^2 \) inversion.

The synthetic seismogram for this model is computed using the ray tracing for an offset range from 0 to 3500 m (Figure 4). Assume that three traveltime parameters are estimated accurately in the velocity analysis with the values \( t_0 = 0.901 \text{ s}, V_{\text{NMO}} = 2235 \text{ m/s}, \) and \( S_2 = 1.054. \) Figure 2 shows the real part of the reflection amplitude \( R(x) \). Reflection amplitude changes polarity between 1500 and 2100 m, and critical reflection is at the offset of 2900 m.

Velocity interpretation is performed for the five three-parameter models and one two-parameter CV model mentioned above. We analyze different P-wave velocity models in the overburden, assuming
that S-wave velocity and density distributions are the same in all cases. The model parameters are computed from inverting travelt ime parameters (equation A-6) for the given value of \( n \). To perform the inversion for the two-parameter CV model, we ignore the value of heterogeneity coefficient \( S_2 \) and use the standard Dix equations. Figure 3 shows kinematically equivalent velocity distributions computed in the inversion. The three-parameter velocity models (\( n = -2, -1,0,1,2 \)) are close to the ideal model (\( n = 1 \)), but the parameters for the CV model have the largest deviation. The error in depth estimate is largest for the CV model (\( \Delta H = 6.9 \) m); the three-parameter velocity models result in an error of less than 0.5 m (Figure 4).

Figure 5 shows the true amplitude correction factor computed from geometrical spreading. The correction factor is the ratio of geometrical spreading calculated for the interpreted velocity model compared to the true model (\( n = 1 \)). When the velocity interpretation is close to the true model, the correction factor is equal to one. The influence of this factor is very small for three-parameter models at near offset and increases at large offsets, where the reflected wave becomes a diving wave. At this offset, the caustic singularity appears and creates a region where geometrical spreading becomes infinite.

Because the position of the caustic singularity is model dependent, its influence on geometrical spreading is also model dependent. For the two-parameter velocity distribution, the deviation of the correction factor dramatically increases with offset. A similar trend in the deviation from the exact velocity model is observed in the offset-angle plot (Figure 6).

Figure 7 shows model-dependent true-amplitude AVO curves computed from the AVO response obtained from the seismogram (Figure 1). Two different effects are evident: the scaling effect from geometrical spreading correction and the stretching effect from offset/incident-angle conversion. These effects are most significant for CV models and are observed for all incident angles. Analysis of Figures 5–7 shows that stretching effect for the three-parameter model is evident for all angles, whereas scaling effect mostly takes place at large incident angles where postcritical reflection is generated. Therefore, the three-parameter velocity models are very similar for the precritical range of incident angles.

**AVA INVERSION**

To quantify the errors imposed by velocity misinterpretation (the wrong choice of velocity model), we solve the AVO inverse problem. Only the parameters of the target layer are estimated. We assume that the velocities in the overburden are computed from the kinematic interpretation described above, and the density for the overburden is known.

We consider two inversion methods: the least-squares method, where the AVO attributes are computed, and the nonlinear Nelder-Mead method (Himmelblau, 1972), with estimation of the medium parameters. The first method is widely used in the industry for conventional AVO inversion where only small incidence angles are involved; the second method is free of these restrictions. We use model-dependent true-amplitude AVO curves shown in Figure 7 as input data in the inversion.

The least-squares method is based on approximation of the coefficient (Aki and Richards, 1980):

\[
R(\theta_H) \approx R(0) + G \sin^2(\theta_H) + K \sin^4(\theta_H),
\]

where \( R(0) \), \( G \), and \( K \) denote the intercept, gradient, and curvature of the reflection coefficient and where \( \theta_H \) is the angle of incidence. This approximation is valid up to 50° in incident angle, so we limit the source-to-receiver distance up to 2500 m to guarantee only precritical reflections.

Figure 8 shows the true-amplitude AVO dependencies plotted against \( \sin^2(\theta_H) \). Stretching and scaling are observed; however, the stretching effect is more evident than the scaling effect. The attributes defined in equation 12 differ, depending on the velocity model; however, intercept \( R(0) \) is very close to the true intercept for all considered curves. The shape of the AVO-dependent curves indicates that using the CV model will produce the largest errors in \( G \) and \( K \).

The errors in AVO attributes are plotted in Figure 9. As expected, the largest errors are obtained for the CV model. Note that the errors in attributes increase with their order, regardless of the velocity model. Intercept estimates are most accurate, gradient has medium accuracy, and curvature has the largest error. It is easy to see that all curves have practically the same AVO intercept, whereas the AVO gradient and AVO curvature are different for the different velocity models. The results from the three-parameter interpretation are more accurate than the two-parameter one. Error values have some sym-
Figure 3. Kinematically equivalent overburden velocity models computed by inverting the traveltime parameters. The red line corresponds to the CV model (the value for heterogeneity coefficient is ignored). Lines in other colors correspond to three-parameter velocity models.

Figure 4. Errors in estimation of depth for different velocity models.

Figure 5. Errors in relative geometrical spreading for different velocity models normalized with the true velocity model ($n = 1$).

Figure 6. Model-dependent offset-to-angle conversion.

Figure 7. Model-dependent true-amplitude AVA curves computed from the original AVO response (Figure 2) using different velocity models. Note the effect of stretching and scaling factors.

Figure 8. Precritical true-amplitude AVA dependencies from Figure 7 plotted versus sine squared of the incidence angle.
Wrong velocity interpretation of estimated traveltime parameters has two major effects on amplitude data: stretching from offset-to-angle conversion and scaling from geometrical spreading correction. Results show that these effects are significant for the two-parameter velocity interpretation even for small offsets, whereas they decrease significantly for three-parameter velocity interpretation and distort mostly amplitudes at far offsets. However, the family of three-parameter velocity models describes a variety of velocity distributions. Geologic information might be useful for estimating the three-parameter models. Among reservoir parameters, P-wave velocity estimates have the best accuracy for all considered models, whereas S-wave velocity estimates have small errors only for the three-parameter models. The largest error from applying the CV model (the least accurate case) is in reservoir density, which is about 12%. The largest error from the application of three-parameter models is in S-wave velocity (about 3%) for the model with \( n = -2 \).

Figure 10a shows the relative errors in the reservoir parameters computed from precritical incident angles only. In general, errors gradually increase with parameter deviation from the true model. The most inaccurate estimates in reservoir parameters are obtained for the CV case, and they are three times larger than for the three-parameter velocity models. Among reservoir parameters, P-wave velocity estimates have the best accuracy for all considered models, whereas S-wave velocity estimates have small errors only for the three-parameter models. The largest error from applying the CV model (the least accurate case) is in reservoir density, which is about 12%. The largest error from the application of three-parameter models is in S-wave velocity (about 3%) for the model with \( n = -2 \).

Figure 10b shows the relative error in reservoir-parameter estimates from AVO inversion; postcritical incident angles are included. We observe a similar trend in parameter estimation for different classes of the velocity models. The most inaccurate estimates are obtained for the two-parameter model, whereas estimates for the three-parameter models become more precise closer to the true model. Comparison with the results for precritical reflections (Figure 10a) shows that including postcritical reflections increases errors in S-wave velocity estimation but improves P-wave velocity and density estimates. This is because distortions from velocity analysis at the postcritical region are more significant than at precritical offsets. When we use only the PP reflection coefficient in inversion, we obtain very good estimates for P-wave velocities but less precise estimates for S-wave velocities. Therefore, it turns out that the accuracy of P-wave velocity does not change dramatically with a three-parameter model change as in the case of S-wave velocity (Figure 10). Different behavior in accuracy among reservoir parameters for two- and three-parameter models in Figure 10 is probably because scaling and stretching effects have the largest influence on AVO dependencies for the two-parameter case, making these results more unstable.

**CONCLUSIONS**
velocity models than for the CV model. Including postcritical offsets improves P-wave velocity and density estimates but impairs S-wave velocity accuracy for three-parameter models; estimates for CV are unstable.

Investigations have been made under the assumption that travel-time parameters are estimated precisely. Therefore, the presence of additional errors in the AVA inversion caused by uncertainties in traveltime-parameter estimation requires further analysis. We expect that three-parameter velocity models will produce better estimates of the reservoir parameters than the two-parameter model.

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APPENDIX A

THE POWER-GRADIENT VELOCITY MODEL

The parametric form of the time-offset relationship for the vertically heterogeneous velocity model \( V(z) \), \( z \in [0, H] \) can be written as the parametric equations

\[
x(p) = 2 \int_0^H \frac{pV(z)dz}{\sqrt{1 - p^2V^2(z)}}
\]

and

\[
t(p) = 2 \int_0^H \frac{dz}{V(z)\sqrt{1 - p^2V^2(z)}},
\]

where \( p \) is the ray parameter or horizontal slowness. Substituting the power-gradient velocity model from the equation 2 into the offset-traveltime equations A-1 results in the analytic expressions given by the hypergeometric functions (Stovas, 2009):

\[
x(p) = \frac{2pV_0Hn(\gamma^{n+1} - 1)}{(\gamma^n - 1)(n + 1)} \times \left[ \begin{array}{c}
F_1\left(\frac{n + 1}{2}, \frac{n + 3}{2}, \frac{p^2V_0^2}{2}y^2\right) \\
F_1\left(\frac{n}{2}, \frac{n + 1}{2}, \frac{p^2V_0^2}{2}y^2\right)
\end{array} \right] \\
+ \frac{3}{8}p^4V_0^4 n + 1 (\gamma^{n+5} - 1) + \cdots
\]

\[
t(p) = \frac{2Hn(\gamma^{n-1} - 1)}{(\gamma^n - 1)(n - 1)V_0} \times \left[ \begin{array}{c}
F_1\left(\frac{n - 1}{2}, \frac{n + 1}{2}, \frac{p^2V_0^2}{2}y^2\right) \\
F_1\left(\frac{n - 1}{2}, \frac{n + 1}{2}, \frac{p^2V_0^2}{2}y^2\right)
\end{array} \right] \\
+ \frac{3}{8}p^4V_0^4 n + 1 (\gamma^{n+3} - 1) + \cdots
\]

To compute the traveltime parameters in vertically heterogeneous media \( V(z) \), \( z \in [0, H] \), we need to define the velocity moments (Fomel and Grechka, 2001):

\[
I_m = \int_0^H V^m(z)dz, \quad m = -1, 1, 3.
\]

Then the traveltime parameters can be expressed by combinations of the velocity moments:

\[
t_0 = 2I_{-1},
\]

\[
V_{NMO}^2 = \frac{I_1}{I_{-1}},
\]

\[
S_2 = \frac{I_2}{I_1},
\]

\[
S_3 = \frac{I_3}{I_1},
\]

\[
\vdots
\]

\[
S_k = \frac{I_{2k-1}}{I_1}.
\]

In a homogeneous medium \( t_0 = 2H/V_0, V_{NMO} = V_0 \) and all heterogeneity coefficients are equal to one, i.e., \( S_1 = 1, k = 2, 3, \ldots \). The expressions for the traveltime parameters in terms of the model parameters can be obtained by substituting equation 2 into equations A-4 and A-5. Explicitly, the traveltime parameters have the following forms (Stovas, 2009):

\[
t_0 = \frac{2H}{V_0(n - 1)} \gamma^{n-1} - 1
\]

\[
V_{NMO}^2 = V_0^2(n - 1) \gamma^{n+1} - 1
\]
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\[ S_2 = \frac{(n + 1)^2}{(n + 3)(n - 1)} \frac{\left(\gamma^n + 3 - 1\right)\left(\gamma^n - 1\right)}{\left(\gamma^n + 1 - 1\right)^2}, \]

\[ S_3 = \frac{(n + 1)^3}{(n + 5)(n - 1)^2} \frac{\left(\gamma^n + 5 - 1\right)\left(\gamma^n - 1\right)^2}{\left(\gamma^n + 1 - 1\right)^3}, \]

\[ \ldots, \]

\[ S_k = \frac{(n + 1)^k}{(n + 1 + 2k)(n - 1)^{k-1}} \frac{\left(\gamma^n + 2k - 1\right)\left(\gamma^n - 1\right)^k}{\left(\gamma^n + 1 - 1\right)^k}. \quad \text{(A-6)} \]

Note that \( \lim_{m \to \infty} \left(\frac{\gamma^m - 1}{m}\right) = \ln \gamma. \)

By substituting equations A-6 into equations A-3, we obtain the parametric moveout expression in terms of traveltimes parameters:

\[
x(p) = p t_0 V_{\text{NMO}}^2 \left[ 1 + \sum_{m=2}^{\infty} q_m S_m (p V_{\text{NMO}})^{2m-2} \right],
\]

\[
t(p) = t_0 \left[ 1 + \sum_{m=2}^{\infty} q_m S_{m-1} (p V_{\text{NMO}})^{2m-2} \right], \quad \text{(A-7)}
\]

where \( q_m = (1 \cdot 3 \cdot \cdots \cdot (2m-3))/(2 \cdot 4 \cdot \cdots \cdot (2m-2)), \) \( m = 2, 3, \ldots, \) and \( S_1 = 1. \) Expanding the traveltimes parameters from equation A-6 in the Taylor series at \( \gamma = 1, \) we obtain

\[
t_0 = \frac{2H}{V_0} \left[ 1 - \frac{1}{2} (\gamma - 1) - \frac{1}{12} (\gamma - 5)(\gamma - 1)^2 + \frac{1}{8} (\gamma - 3) \times (\gamma - 1)^3 + \cdots \right].
\]

\[
V_{\text{NMO}}^2 = V_0^2 \left[ 1 - (\gamma - 1) + \frac{1}{6} \ln (\gamma - 1)^2 + 0 + \cdots \right].
\]

\[
S_2 = 1 + \frac{1}{3} (\gamma - 1)^2 + \frac{1}{3} (\gamma - 1)^3
\]

\[
+ \frac{1}{60} (n^2 + 2n - 20)(\gamma - 1)^4 + \cdots,
\]

\[
S_3 = 1 + (\gamma - 1)^2 + (\gamma - 1)^3
\]

\[
+ \frac{1}{60} (3n^2 + 10n - 72)(\gamma - 1)^4 + \cdots,
\]

\[
\ldots,
\]

\[
S_k = 1 + \frac{k}{6} (k - 1)(\gamma - 1)^2 + \frac{k}{6} (k - 1)(\gamma - 1)^3
\]

\[
+ \frac{1}{360} k(k - 1)(54 + 3k^2 + 2n - 3n^2 - 3k - 4kn)
\times (\gamma - 1)^4 + \cdots. \quad \text{(A-8)}
\]

Heterogeneous coefficients through the heterogeneous coefficient of the second-order \( S_2. \) Then we can analyze the influence of \( n. \) Inverting the series for the second-order heterogeneous coefficient from equation A-8, we obtain the series for the velocity contrast \( \gamma \) as follows:

\[
\gamma = 1 + \sqrt[3]{3}(S_2 - 1)^{1/2} + \frac{3}{2}(S_2 - 1)
\]

\[
+ \frac{5 + 3n^2 + n^2}{40} \sqrt[3]{3}(S_2 - 1)^{1/2} + \cdots. \quad \text{(A-9)}
\]

Note that \( n \) appears in the third-order coefficient of the series. Using equation A-9, we can express the higher-order heterogeneous coefficients from equation A-8 in terms of the second-order coefficient (see equation 3). Substituting series A-9 into the second expression in equation A-8 results in the series for parameter \( \lambda = V_0/V_{\text{NMO}} \) (see equation 7).

**APPENDIX B**

**SERIES FOR RELATIVE GEOMETRICAL SPREADING**

Let us introduce the normalized offset as a function of slowness

\[
\bar{x}(p) = \frac{x(p)}{V_{\text{NMO}}}, \quad \text{(B-1)}
\]

Substituting the expression for offset from equation A-7 into equation B-1 and performing the inversion of the series, we obtain the series for slowness in terms of normalized offset:

\[
p(\bar{x}) = \frac{1}{V_{\text{NMO}}^2} \left[ \bar{x} - q_2 S_2 \bar{x}^3 + (3 q_2 S_2^2 - q_3 S_3) \bar{x}^5 + \cdots \right]. \quad \text{(B-2)}
\]

Substituting equation B-2 into the equation 11 for the radiation pattern results in

\[
\cos(\theta_0) = \sqrt{1 - p(\bar{x})^2} = \sqrt{1 - \lambda^2 \left[ \bar{x}^2 - S_2 \bar{x}^4 + \frac{7 S_2^2 - 3 S_3}{4} \bar{x}^6 + \cdots \right]}, \quad \text{(B-3)}
\]

where \( \lambda = V_0/V_{\text{NMO}}. \) Expansion of equation B-3 into Taylor series gives

\[
\cos(\theta_0) = 1 - \frac{1}{2} \lambda^2 \bar{x}^2 - \frac{\lambda^4}{8} \bar{x}^4
\]

\[
- \frac{14 S_2^2 \lambda^2}{16} - 6 S_3 \lambda^2 - 4 A \lambda^4 S_2 + \lambda^6 \bar{x}^6 + \cdots. \quad \text{(B-4)}
\]

Substituting equation B-2 into the relation for the incident angle from equation 5 at the target level results in

\[
\sin(\theta_H) = p V(H) = \frac{\sin(\theta_0)}{V_0} V(H) = \gamma \lambda
\]

\[
\times \left[ \bar{x} - \frac{1}{2} S_2 \bar{x}^3 + \frac{3}{4} \left( \frac{1}{2} S_2 - \frac{1}{2} S_3 \right) \bar{x}^5 + \cdots \right]. \quad \text{(B-5)}
\]
The second term in equation 11 for relative geometrical spreading can be written similarly. By using the expression for the offset from equation A-3, expanding the second term into the Taylor series, and substituting the equation B-2, we obtain

\[
\left[ \frac{x \, dx}{p \, dp} \right]^{1/2} = V_{\text{NMO}} \left[ 1 + S_2 p^2 V_{\text{NMO}}^2 + \frac{1}{8} (9S_3 - S_2^2) p^4 V_{\text{NMO}}^4 + \cdots \right] \\
= V_{\text{NMO}} \left[ 1 + S_2 \tilde{x}^2 + \frac{9}{8} (S_3 - S_2^2) \tilde{x}^4 + \cdots \right].
\]  

(B-6)

REFERENCES


