Kinematically equivalent velocity distributions

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ABSTRACT

For a layered medium, the seismic velocity model can be vertically heterogeneous within the layers. The travelt ime parameters estimated from each reflection must be converted into layer travelt ime parameters by using the layer-stripping method. The layer travelt ime parameters must be inverted into layer velocity model parameters. Interpretation or inversion of layer travelt ime parameters depends on the chosen velocity model within the layer. Different or kinematically equivalent velocity distributions can result in the same travel- time parameters. The inversion problem for travelt ime parameters is strongly nonunique even if they are estimated accurately. To evaluate the accuracy of a velocity model, one can choose the phase for the two-way propagator. The discrepancy in this phase factor between the kinematically equivalent velocity models depends on the number of travelt ime parameters estimated and increases with spatial frequency. By estimating two travelt ime parameters, we approximately preserve the average velocity, regardless of the complexity of the vertically heterogeneous model. By estimating three travelt ime parameters, we approximately preserve the average velocity gradient.

INTRODUCTION

One of the crucial challenges in seismic studies is inverting the travelt ime parameters estimated in velocity analysis into the model parameters. Practical problems arise with this procedure. Small uncertainties in estimating travelt ime parameters can result in significant uncertainties in the model parameters (Hajnal and Sereda, 1981), especially for relatively thin layers. Travelt ime parameters are influenced by seismic noise, offset spread, and the trade-off between them. Being computed for each reflector, the travelt ime parameters must be converted into the travelt ime parameters for each layer using a Dix procedure (Stovas and Ursin, 2007). Next, the travelt ime parameters for each layer must be inverted into model parameters for each layer. Standard Dix (1955) equations deal with two travelt ime parameters (two-way vertical travelt ime and NMO velocity) and imply the constant-velocity isotropic model, a two-parame- ter model. With this assumption, the velocity computed by the Dix equation is referred to as interval velocity and the nonhyperbolicity of reflection travelt ime is induced. This means our model is intrinsi- cally hyperbolic.

However, if we consider a more complicated, vertically heterogeneous model for a given layer, the velocity computed by the Dix equation is termed the NMO velocity for this layer, and we must also compute the intrinsic parameters responsible for nonhyperbolicity. With a standard approach, the problem is unique. Two travelt ime parameters correspond to only two model parameters. With a vertically heterogeneous layer and more than two layer travelt ime parameters estimated, we must choose a specific model with parameters that we need to compute. The inversion becomes nonunique. Even if we assign the model with the same number of parameters as estimated from traveltimes, we have a family of velocity distributions resulting in the same set of travelt ime parameters. Some practical considerations of nonuniqueness in estimating velocity versus depth function from reflection travelt ime are pointed out by Al-Chalabi (1997).

In this paper, I consider the situation if the number of the velocity model parameters is larger than the number of estimated travelt ime parameters, thereby creating a family of kinematically equivalent velocity distributions that give the same travelt ime parameters. The deviation in the phase of the two-way propagator for this family of velocity distributions can be used as a criterion for accuracy in the velocity model.

I illustrate this approach by applying a three-parameter velocity model for two travelt ime-parameter estimates and a four-parameter velocity model for three travelt ime-parameter estimates. The linear velocity model is used as the three-parameter model, and the modified Faust (1951) model is used as the four-parameter model. I use analytical models for all computations to avoid numerical errors. (Some details concerning the analytical models are omitted for brevity.) The inversion of layer travelt ime parameters into layer velocity model parameters is nonunique and results in two solutions: with a
positive and a negative velocity gradient (Stovas and Ursin, 2007).

Estimation errors in the traveltime parameters are ignored because of noise and data sampling, which also result in uncertainties in the layer parameters (Bickel, 1990; Lines, 1992; Kosloff and Sudman, 2001). The phase factor in the propagator is used to evaluate the seismic image’s accuracy because of the limited number of estimated traveltime parameters. Another way to link the accuracy in velocity to accuracy in structure is the velocity propagation method (Fomel and Landa, 2005).

First, I discuss parameterization of reflection traveltime versus parameterization in the velocity model. Then the two-, three-, and four parameter models are defined by computing the traveltime parameters and the phase factor. The average velocity and average velocity gradient are computed for each velocity model. In numerical examples, I show the results of interpreting a three-parameter velocity model with two estimated traveltime parameters and interpreting a four-parameter velocity model with three estimated traveltime parameters. The deviation in the phase factor for the propagator illustrates the accuracy of the velocity models. In the appendices, I show how to derive the traveltime parameters from a vertically heterogeneous medium, the continuous version of the Dix equations, and a derivation of the phase factor for a two-way propagator from two vertically heterogeneous models.

**HOW MANY TRAVELTIME PARAMETERS DO WE NEED?**

The traveltime parameters are defined from the series expansion of reflection traveltime squared with offset \( x \) (Taner and Koehler, 1969):

\[
t^2(x) = t_0^2 + \frac{x^2}{V_{nmo}^2} + \frac{(1 - S_2)x^4}{4V_{nmo}^4} + \frac{(2S_2 - S_2 - S_3)x^6}{8V_{nmo}^6} + \ldots,
\]

where \( t_0 \) is two-way vertical traveltime, \( V_{nmo} \) is the NMO velocity, and \( S_2 \) and \( S_3 \) are heterogeneity coefficients of the second and third order, respectively (Fomel and Grechka, 2001). These kinematic parameters are defined in Appendix A. The coefficients (traveltime parameters) in expansion 1 are estimated in velocity analysis from each reflection. By using the recursive Dix inversion, we can compute these parameters for each layer. The next step is to invert the traveltime parameters into the two model parameters: layer thickness \( H \) and velocity distribution within the layer \( V(z) \).

In Appendix B, we learn that if we would estimate the NMO velocity versus two-way vertical traveltime \( V_{nmo}(t_0) \) as a continuous function, we could compute the velocity model \( V(z) \) accurately by using the continuous version of the Dix equation. The problem is that we estimate \( V_{nmo}(t_0) \) for a given set of \( t_0 \) — namely, for the top and the bottom of each layer. For that case, to recover the velocity model between the top and bottom of the layer, we need to estimate more traveltime parameters.

The number of traveltime parameters estimated in velocity analysis depends on the velocity model used for inversion. For the constant velocity model with two model parameters, layer thickness and interval velocity, we need only two estimated traveltime parameters: two-way vertical traveltime and NMO velocity. Standard Dix (1955) inversion manipulates two traveltime parameters and presumes a two-parameter velocity model. If the velocity model is more complicated, more traveltime parameters are required for inversion. In practice, the maximum number of estimated traveltime parameters is three.

If the accurate velocity model requires more parameters then we have estimated from the seismic reflection, we cannot resolve all of the velocity model parameters. In this case, there is a family of different velocity distributions that result in the same traveltime parameters we estimated. These velocity distributions are kinematically equivalent with respect to a given number of traveltime parameters. The number of estimated traveltime parameters gives the order of equivalency. The smaller the order of equivalency or the fewer traveltime parameters we estimate, the broader the family of kinematically equivalent velocity distributions.

To illustrate the effect of the limited number of estimated traveltime parameters, I use the phase factor for the propagator, which differs for various models. By computing the deviation in this phase factor between the two-parameter model and the family of kinematically equivalent three-parameter models, we can evaluate the maximum horizontal wavenumber or spatial frequency applicable for the migration. I illustrate this idea with two-, three-, and four-parameter models. In this analysis, we presume that all traveltime parameters are estimated correctly from the seismic data, although in practice the accuracy of traveltime parameters reduces as their order increases (e.g., noise, trade-off).

**TWO-PARAMETER VELOCITY MODEL**

This is the simplest velocity model defined by the thickness \( H \) and the constant velocity \( V_0 \):

\[
V(z) = V_0.
\]

By applying the standard Dix (1955) equations to reflection traveltime with short offset spread, the layer traveltime parameters are the two-way vertical traveltime \( t_0 \) and the NMO velocity \( V_{nmo} \), which we estimate by using the hyperbolic moveout or only two terms in equation 1. In this case, the heterogeneity coefficients responsible for nonhyperbolicity of reflection traveltime are \( S_j = 1, k = 2, 3, \ldots \).

To imply this velocity model, simple equations are required for the interpretation:

\[
t_0 = \frac{2H}{V_0},
\]

\[
V_{nmo} = V_0.
\]

**THREE-PARAMETER VELOCITY MODEL WITH TWO ESTIMATED TRAVELTIME PARAMETERS**

There are many analytical three-parameter velocity models. In this analysis, I choose the linear velocity model because it is simple and computations can be done analytically.

The linear velocity model, given by

\[
V(z) = V_0(1 + \beta z),
\]

requires three parameters: thickness \( H \), velocity to the top \( V_0 \), and velocity contrast \( \gamma = 1 + \beta H \), which is the ratio between velocity to the bottom and velocity to the top. To define the model parameters,
three independent traveltime parameters must be estimated (Appendix A):

\[ t_0 = \frac{2H}{V_0} \ln \gamma, \]

\[ V_{\text{nmo}}^2 = \frac{V_0^2 \gamma^2 - 1}{2 \ln \gamma}, \]

\[ S_2 = \frac{(\gamma^2 + 1)}{(\gamma^2 - 1)} \ln \gamma. \] (5)

The traveltime curve is nonhyperbolic, and the heterogeneity coefficient \( S_1 > 1 \), regardless of whether the velocity increases or decreases with depth. Inverting equations 5 leads to two solutions, one with \( \gamma > 1 \) and another with \( \gamma < 1 \) (Stovas and Ursin, 2007).

If only two traveltime parameters are available for interpretation, we have a family of linear velocity distributions that are kinematically equivalent to the second order of equivalency. In this case, from equations 5 we have

\[ H = V_{\text{nmo}} t_0 \sqrt{\frac{(\gamma - 1)}{2(\gamma + 1)\ln \gamma}}, \]

\[ V_0 = V_{\text{nmo}} \sqrt{\frac{2 \ln \gamma}{\gamma^2 - 1}}, \] (6)

with \( \gamma \) now a free parameter.

For propagating or nonvanishing waves \( |k_x| < 2|\omega|/V(H) \), the phase factor for the linear velocity model is given in Appendix C. To quantify the discrepancy between the constant velocity model and the family of linear velocity models, I compute the difference in the phase factors. The phase factors for linear velocity models are subtracted from the phase factor computed for the constant velocity model (equation C-3), with interpretation given in equation 3:

\[ \varphi_2(\omega, k_x) = \frac{2H|\omega|}{V_0} \sqrt{1 - \frac{k_x^2 V_0^2}{4|\omega|^2}}. \] (7)

FIVE-PARAMETER VELOCITY MODEL WITH THREE ESTIMATED TRAVELTIME PARAMETERS

A similar approach can be used to define the kinematically equivalent velocity distributions from three traveltime parameters. From the Dix equations, we can estimate three traveltime parameters using a nonhyperbolic traveltime approximation.

To introduce the four-parameter velocity model, I use the modified Faust (1951) model:

\[ V'(z) = V_0(1 + \beta z). \] (8)

For given parameters \( n \), the velocity model defined in equation 8 is reduced to well-known velocity models: linear velocity \( (n = 1) \), exponential velocity \( (n = 0) \), linear slowness \( (n = -1) \), and linear slope \( (n = -2) \).

Compared with the linear velocity model (equation 4), it has one additional parameter: \( n \). The velocity gradient changes with depth and can be computed by extending equation \( V(z) = V_0(1 + \beta z)^{1/n} \) in series with respect to depth.

Four independent traveltime parameters are required to define this model (Appendix A):

\[ t_0 = \frac{2H}{V_0} \frac{n}{(n - 1)} \frac{\gamma^{n-1} - 1}{\gamma^{n} - 1}, \]

\[ V_{\text{nmo}}^2 = \frac{V_0^2 (n - 1)}{2} \frac{\gamma^{n+1} - 1}{\gamma^{n} - 1}, \]

\[ S_2 = \frac{(n + 1)^2}{(n + 3)(n - 1)} \frac{(\gamma^{n+3} - 1)(\gamma^{n-1} - 1)}{(\gamma^{n+1} - 1)^2}, \]

\[ S_3 = \frac{(n + 1)^3}{(n + 5)(n - 1)^2} \frac{(\gamma^{n+5} - 1)(\gamma^{n-1} - 1)^2}{(\gamma^{n+1} - 1)^3}. \] (9)

The computation of traveltime parameters given in equation 9 for some values of \( n \) requires taking the limit of the sort: \( \lim_{n \to \infty} (\gamma^n - 1)/|m| = \ln \gamma \).

Note that the coefficients \( S_1 \) and \( S_2 \) reach maximum values at \( n = -1 \), which corresponds to the linear slowness model:

\[ \max_n S_2 = S_2(n = -1) = \frac{(\gamma^2 - 1)^2}{4\gamma^2 \ln^2 \gamma}, \]

\[ \max_n S_3 = S_3(n = -1) = \frac{(\gamma^2 - 1)^3(\gamma^2 + 1)}{16\gamma^4 \ln^3 \gamma}. \] (10)

If only three traveltime parameters are available for interpretation, we have a family of high-order velocity distributions that are kinematically equivalent to the third order of equivalency. If we compare this model with the linear velocity one, \( n \) is a free parameter. For propagating waves, the phase factor for the nonlinear velocity model is given in Appendix C.

AVERAGE VELOCITY AND AVERAGE VELOCITY GRADIENT

It is interesting to investigate how accurately we preserve the average velocity by using two-, three-, and four-traveltime parameter models and accurately we preserve the average velocity gradient by using three- and four-traveltime parameter models.

The average velocity is defined as

\[ \overline{V}_1 = \frac{2H}{t_0}. \] (11)

For the constant velocity model described in equation 2, the average velocity is equal to NMO velocity:

\[ \overline{V}_2 = V_{\text{nmo}}. \] (12)

The subscript in average velocity notation shows the number of traveltime parameters used for computation.

For the linear velocity model given in equation 4, we must substitute expressions for thickness and vertical traveltime from equation 5 into equation 11:

\[ \overline{V}_3 = V_{\text{nmo}} \sqrt{\frac{2(\gamma - 1)}{(\gamma + 1)\ln \gamma}}. \] (13)

For the high-order velocity model from equation 8, we must substitute expressions for thickness and vertical traveltime from equation 9 into equation 11:
The average velocity gradient can be defined by the heterogeneity coefficient \( S^2 \) using corresponding equations 5 and 9. This definition of the average velocity gradient is intuitive rather than mathematical. Although the average velocity in equation 11 can be defined by the ratio of velocity moments \( I_0/I_1 \) (Appendix A), the average velocity gradient is just the interpretation of the \( S^2 \) in terms of the velocity contrast using the corresponding expression from the linear velocity model (equation 5).

**NUMERICAL EXAMPLES**

Let us assume that we have correctly estimated the first two traveltime parameters, \( t_0 = 0.5 \) s and \( V_{nmo} = 2.8 \) km/s. If we interpret them within the framework of the linear velocity model, the third traveltime parameter \( S^2 \) could be arbitrary. In Figure 1, we show the family of kinematically equivalent velocity distributions that have the same vertical two-way traveltime and NMO velocity but different \( S^2 \), which is defined by the velocity contrast \( \gamma \). In this computation, the maximum velocity contrast \( \gamma_{max} = 1.6 \). Each velocity distribution shown in Figure 1 has a different layer thickness and average velocity. The maximum relative error in thickness is about 0.2%, and the maximum relative error in average velocity is about 0.9%.

The discrepancy between the velocity models can be illustrated by deviations in the phase factor \( \varphi \). In Figure 2, one can see the deviation in the phase factor \( \Delta \varphi = \varphi_1 - \varphi_2 \) computed for the temporal frequency of 40 Hz and plotted against the horizontal wavenumber \( k_x \). Note that different velocity distributions have different maximum horizontal wavenumbers for propagating waves. If we set the maximum value for deviation in phase factor as \( \pi/4 \), we derive the limited horizontal wavenumber, which is smaller than the maximum horizontal wavenumber from the constant velocity model. It means the dipping reflections below the layer of interest could be distorted by improper interpretation with a two-traveltime-parameter estimation.

Let us assume that we also estimated the third traveltime parameter, \( S^2 = 1.054 \). Different values of parameter \( n \) give different velocities to the top (Figure 3) and different layer thicknesses (Figure 4).
From equation 10, we can evaluate the minimum velocity contrast $\gamma_{\text{min}} = 1.4946$, which corresponds to the estimated value for the heterogeneity coefficient. Therefore, the dashed isoline $S_z = 1.054$ does not cross the solid curved line $(V_{\text{rms}}/V_0)^2$ in Figure 3 with $\gamma = 1.4$, and the dashed isoline $S_z = 1.054$ does not cross the solid curved line $(V_{\text{rms}}/2H)$ in Figure 4 with $\gamma = 1.4$. In Figure 5, one can see the velocity distributions based on the model in equation 9, whose first three travelt ime parameters are the same. All of the velocity distributions have approximately the same layer thickness, average velocity, and velocity-to-depth gradient. In Figure 6, we present the deviation in phase factor $\Delta \varphi = \varphi_2 - \varphi_1$ computed for the same temporal frequency of 40 Hz and plotted against the horizontal wavenumber $k_x$. Compared with the previous case, the deviations are much smaller, but they could have a different sign (for positive and negative values of $n$). Note also that the distributions with negative $n$ give a limited horizontal wavenumber smaller than the linear velocity model.

To illustrate how the average velocity and average velocity gradient are preserved in two-, three-, and four-parameter models, we plot the average velocity against the velocity contrast using equations 12–14 (Figure 7a, to the top). For the three-parameter model, the average velocity depends on the velocity contrast; for the four-parameter model, it depends on the velocity contrast and $n$. Even for large values of velocity contrast, the variations in the average velocity are insignificant. One can see that by estimating two travelt ime parameters, we approximately preserve the average velocity.

The average velocity gradient can be characterized by the heterogeneity coefficient $S_z$. In Figure 7b (to the bottom), we plot $S_z$ against $\gamma$ for a three-parameter model from equation 5 and for a four-parameter model from equation 11 with $n \in \{-4, -2, 2, 4\}$. The difference between $S_z$ from three- and four-parameter models characterizes the difference in the average velocity gradient between these models. By estimating three travelt ime parameters, we approximately preserve the average velocity gradient.

Figure 5. Kinematically equivalent velocity distributions with three travelt ime parameters. The linear velocity model is shown by the solid black line, and the high-order models are shown by gray lines ($n = 1, 2, 4$ corresponds to solid, dashed, and dotted lines, respectively) and light gray lines ($n = -1, -2, -4$ corresponds to solid, dashed, and dotted lines, respectively).

Figure 6. Deviation in the phase factor between the linear velocity model and the family of the high-order velocity models. The temporal frequency is 40 Hz. The phase errors are relatively small, and the only limitation is from the maximum applicable horizontal wave number corresponding to evanescent waves. Gray lines correspond to positive $n$, and light gray lines correspond to negative $n$ with the same legend as for Figure 5.

Figure 7. (a) The average velocity to the top computed from different velocity models versus the velocity contrast. The average velocity from the constant velocity model $V_0 = V_{\text{rms}} = 2.8$ km/s. (b) The average velocity contrast to the bottom given by the heterogeneity coefficient versus the velocity contrast.
CONCLUSIONS

I define kinematically equivalent velocity distributions, which depend on the number of estimated traveltime parameters. With two estimated traveltime parameters, we guarantee relatively correct layer thickness and average velocity. With three estimated traveltime parameters, we guarantee relatively correct layer thickness, average velocity, and average velocity gradient. Despite that, the critical horizontal wavenumber can be altered significantly.

The accuracy of a velocity model is strongly dependent on the number of traveltime parameters used, even if those parameters are estimated correctly. This affects the propagator phase and can result in significant deviation for large values of the horizontal wavenumber. I illustrate this problem by introducing the deviation in the phase within the family of kinematically equivalent velocity distributions.

By considering two-, three-, and four-parameters models, I show that estimating three traveltime parameters significantly reduces the phase error in the propagator; however, estimating four traveltime parameters only slightly improves the accuracy in the propagator phase.

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APPENDIX A

TRAVELTIME PARAMETERS IN A VERTICALLY HETEROGENEOUS MEDIUM

To compute the traveltime parameters in a vertically heterogeneous medium $V(z), z \in [0, H]$, we need the following integrals (Fomel and Grechka, 2001):

$$I_m = \int_0^H V^m(z)dz, \quad m = -1, 1, 3, \ldots, \ldots$$  \hspace{1cm} (A-1)

The traveltime parameters are given by combinations of these integrals:

$$t_0 = 2I_{-1},$$

$$V_{nmo}^2 = \frac{I_1}{I_{-1}},$$

$$S_2 = \frac{I_1}{I_1},$$

$$S_3 = \frac{I_1}{I_1},$$

$$\vdots$$

$$S_k = \frac{I_{2k-1}}{I_{2k-1}}.$$  \hspace{1cm} (A-2)

In a homogeneous medium $V(z) = V_0$, all heterogeneity coefficients are equal to one: $S_k = 1, k = 2, 3, \ldots$.

APPENDIX B

ORIGIN OF THE DIX EQUATION

In a vertically heterogeneous medium $V = V(z)$, the traveltime parameters for two-way vertical traveltime $t_0$ and NMO velocity $V_{nmo}$ can be computed as shown in Appendix A:

$$t_0(z) = 2 \int_0^z V^{-1}(\xi)d\xi,$$

$$V_{nmo}^2(z) = \frac{\int_0^z V(\xi)d\xi}{\int_0^z V^{-1}(\xi)d\xi}.$$  \hspace{1cm} (B-1)

By composing the product $V_{nmo}^2J_0$, we obtain

$$V_{nmo}^2J_0 = 2 \left[ \int_0^z V(\xi)d\xi \right].$$  \hspace{1cm} (B-2)

Taking the derivative of equation B-2 results in

$$\frac{d(V_{nmo}^2J_0)}{dt_0} = \frac{dV_{nmo}^2}{dz} = V^2(z).$$  \hspace{1cm} (B-3)

Equation B-3 written in the discretized form is known as the Dix equation.

For higher-order traveltime parameters, similar considerations can be used. The heterogeneity coefficient $S_k$ can be computed as (see Appendix A)

$$S_2 = \left[ \begin{array}{cc} \int_0^z V^2(\xi)d\xi \\ \int_0^z V^{-1}(\xi)d\xi \end{array} \right]^2,$$  \hspace{1cm} (B-4)

By composing the product $S_2V_{nmo}^4J_0$ from equations B-1 and B-4, we obtain

$$S_2V_{nmo}^4J_0 = 2 \left[ \int_0^z V^3(\xi)d\xi \right].$$  \hspace{1cm} (B-5)

Taking the derivative over $t_0$ results in
which can be considered the higher-order Dix equation. The general Dix equation for high-order heterogeneity coefficients is given by
\[
\frac{d(Sky_{\text{nm}})}{dt_0} = V^{2k}(z). \tag{B-7}
\]

APPENDIX C

PROPAGATOR FOR VERTICALLY HETEROGENEOUS VELOCITY MODELS

To evaluate accuracy in velocity distributions, I compute the two-way propagator \( M = \exp[i\varphi(\omega, k)] \) for a given temporal frequency \( \omega \) and horizontal wavenumber \( k \). The phase factor for a propagator in a vertically heterogeneous layer of thickness \( H \) is given by
\[
\varphi(\omega, k) = \int_0^H \sqrt{\frac{4\omega^2}{V^2(z)} - k^2} dz. \tag{C-1}
\]

Here, I consider propagating waves only, taking into account that for all models, velocity increases with depth as \( k < 2|\omega|/V(H) \).

Expanding equation C-1 in a Taylor series and using the traveltime parameters defined in equations A-2 results in
\[
\varphi(\omega, k) = \left| \omega \right| t_0 \left[ \sqrt{1 - \frac{k^2 V_{\text{nm}}^2}{4\omega^2}} - \frac{(S_2 - 1) k^4 V_{\text{nm}}^4}{128 \omega^4} - \frac{(S_3 - 1) k^6 V_{\text{nm}}^6}{1024 \omega^6} + \ldots \right]. \tag{C-2}
\]

One can see that the more traveltime parameters we preserve in velocity analysis, the more precisely we estimate the phase factor for large values of horizontal wavenumbers.

For two-, three-, and four-parameter models defined in the main text, I compute the analytical expression for the phase factor \( \varphi(\omega, k) \), where \( j \) denotes the number of model parameters. For the constant velocity model \( V(z) = V_0 \), the phase factor C-1 has a very simple form:
\[
\varphi_2(\omega, k) = \frac{2H|\omega|}{V_0} \sqrt{1 - \frac{k^2 V_0^2}{4\omega^2}}. \tag{C-3}
\]

For the linear velocity model \( V(z) = V_0(1 + \beta z) \), the phase factor is given by
\[
\varphi_3(\omega, k) = \frac{2H|\omega|}{V_0(\gamma - 1)} \left[ \ln \left( \frac{\gamma + \sqrt{1 - \frac{k^2 V_0^2}{4\omega^2}}}{\frac{k_0^2 V_0^2}{4\omega^2}} \right) \right]
+ \sqrt{1 - \frac{k_0^2 V_0^2}{4\omega^2}} - \sqrt{1 - \frac{k^2 V_0^2}{4\omega^2}}. \tag{C-4}
\]

For the nonlinear velocity model \( V(z) = V_0(1 + \beta z) \), the phase factor in the propagator is given by a hypergeometric function:
\[
\varphi_4(\omega, k, n) = \frac{2H|\omega|n}{V_0(\gamma^n - 1)(n - 1)} \times \left[ \frac{\gamma^{n-1}}{2} \right]^n \times \left( \frac{\left( n - \frac{1}{2} \right)}{\gamma^n} \right) \tag{C-5}
- \text{Hypergeometric 2F1}
- \frac{n - \frac{1}{2}}{\left( n - \frac{1}{2} \right)} \times \left( \frac{k_0^2 V_0^2}{4\omega^2} \right) \tag{C-5}
\]

For given \( n \), the hypergeometric function in equation C-5 reduces to well-known elementary transcendental functions.

To quantify the deviation within the family of kinematically equivalent velocity distributions, we use the differences in phase factor: \( \varphi_4(\omega, k, \gamma) - \varphi_3(\omega, k) \), with \( \gamma \) the free parameter, and \( \varphi_4(\omega, k, n) - \varphi_3(\omega, k) \), with \( n \) the free parameter.

REFERENCES

Fomel, S., and E. Land, 2005, Estimating structural uncertainty of seismic images: Presented at the 75th Annual International Meeting, SEG.