Paralinearization of free boundary problems in fluid dynamics

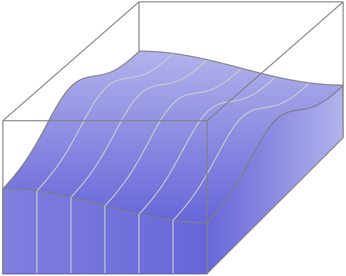
Thomas Alazard

CNRS & Ecole normale supérieure Paris-Saclay

The Abel Symposium
Partial Differential Equations
Waves, Nonlinearities and Nonlocalities

June 13-16, 2023

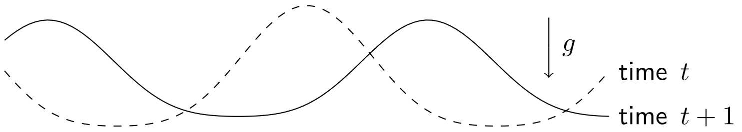
Evolution of the interface air/incompressible fluid



Water wave problem, Hele-Shaw and Muskat equations

Dynamics of an incompressible, irrotational liquid flow

- moving under the force of gravitation
- in a time-dependent domain with a free boundary



Many equations, many different asymptotic regimes (NLS, KdV, BO, P.M.,...)

Full model : free boundary problem

Dispersive (Euler's equations) or parabolic (Darcy's law) equations

Related tools based on paradifferential analysis

Loosely speaking : we're looking for a method to transform the equations to conjugate them to simpler equations.

Consider a time-dependent domain

$$\Omega(t) = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}\,;\, y < \eta(t,x)\} \qquad (d \geq 1)$$

$$\Sigma(t) = \partial\Omega(t) = \{y = \eta(t,x)\}$$

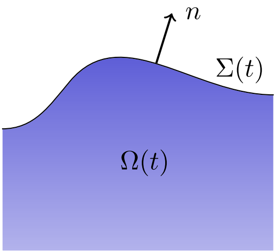
The free surface $\Sigma(t)$ evolves according to

$$\boxed{V_n = u \cdot n}$$

where $u\colon \Omega \to \mathbb{R}^{d+1}$ is the fluid velocity and

$$n = \frac{1}{\sqrt{1+|\nabla\eta|^2}} \begin{pmatrix} -\nabla\eta \\ 1 \end{pmatrix}$$

$$V_n = n \cdot \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ \eta(t,x) \end{pmatrix} = \frac{\partial_t \eta}{\sqrt{1+|\nabla\eta|^2}}.$$



$$\partial_t \eta = \sqrt{1+|\nabla\eta|^2}\, u \cdot n.$$

Reduction to the boundary. Assume

$$\mathrm{curl}_{x,y}\, u = 0 \quad , \quad \mathrm{div}_{x,y}\, u = 0.$$

Then $u = \nabla_{x,y}\phi$ for some ϕ s.t. $\Delta_{x,y}\phi = 0$. Now, set [Zakharov]

$$\psi(t,x) = \phi(t,x,\eta(t,x))$$

and introduce [Craig-Sulem, Lannes] the Dirichlet-to-Neumann operator by

$$G(\eta)\psi = \partial_y\phi - \nabla\eta\cdot\nabla\phi\Big|_{y=\eta} = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\Big|_{y=\eta}.$$

Then

$$\begin{aligned}\partial_t\eta &= \sqrt{1+|\nabla\eta|^2}\,u\cdot n\\ &= \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\\ &= G(\eta)\psi\end{aligned}$$

Reduction to the boundary. Assume

$$\operatorname{curl}_{x,y} u = 0 \quad , \quad \operatorname{div}_{x,y} u = 0.$$

Then $u = \nabla_{x,y}\phi$ for some ϕ s.t. $\Delta_{x,y}\phi = 0$. Now, set [Zakharov]

$$\psi(t,x) = \phi(t,x,\eta(t,x))$$

and introduce [Craig-Sulem, Lannes] the Dirichlet-to-Neumann operator by

$$G(\eta)\psi = \partial_y\phi - \nabla\eta\cdot\nabla\phi\Big|_{y=\eta} = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\Big|_{y=\eta}.$$

Then

$$\boxed{\partial_t\eta = G(\eta)\psi}$$

Need also an equation for ψ. The most simple one is

$$\boxed{\psi = -\eta}$$

which gives the Hele-Shaw equation: $\partial_t\eta + G(\eta)\eta = 0$.

Remark: $i)$ physical equation associated to Darcy's law $u = -\nabla_{x,y}(P+gy)$.

$ii)$ with $\psi + K(\eta)\psi = -\eta$ we get the Muskat equation.

The most beautiful equation for ψ is given by

Theorem [Zakharov, 1966]. Consider an irrotational velocity field $u = \nabla_{x,y}\phi$ satisfying $\partial_t u + u \cdot \nabla_{x,y} u = -\nabla_{x,y}(P + gy)$. Then η and ψ are conjugated:

$$\boxed{\begin{aligned} \frac{\partial \eta}{\partial t} &= \frac{\delta \mathcal{H}}{\delta \psi} \\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathcal{H}}{\delta \eta} \end{aligned}}$$

$$\psi(t,x) = \phi(t,x,\eta(t,x))$$

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^d} \psi G(\eta)\psi \, \mathrm{d}x + \frac{g}{2}\int_{\mathbb{R}^d} \eta^2 \, \mathrm{d}x.$$

(Brenier related the Hele-Shaw and WW problems by a quadratic change of time.)

The most beautiful equation for ψ is given by

Theorem [Zakharov, 1966]. Consider an irrotational velocity field $u = \nabla_{x,y}\phi$ satisfying $\partial_t u + u \cdot \nabla_{x,y} u = -\nabla_{x,y}(P + gy)$. Then η and ψ are conjugated:

$$\boxed{\begin{aligned} \frac{\partial \eta}{\partial t} &= \quad \frac{\delta \mathcal{H}}{\delta \psi} \\ \frac{\partial \psi}{\partial t} &= -\frac{\delta \mathcal{H}}{\delta \eta} \end{aligned}} \qquad \psi(t,x) = \phi(t,x,\eta(t,x))$$

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^d} \psi G(\eta)\psi \,\mathrm{d}x + \frac{g}{2}\int_{\mathbb{R}^d} \eta^2 \,\mathrm{d}x.$$

(Brenier related the Hele-Shaw and WW problems by a quadratic change of time.)

A popular form of the equations:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta + \frac{1}{2}\left|\nabla\psi\right|^2 - \frac{1}{2}\frac{\left(\nabla\eta\cdot\nabla\psi + G(\eta)\psi\right)^2}{1+|\nabla\eta|^2} = 0 \end{cases}$$

Prop (A-Burq-Zuily). This system is equivalent to Euler with free surface.

- not PDE ($G(\eta)$ is a **nonlocal** operator)
- **fully nonlinear** (instead of semi-linear, see [Said 2020])
- the Hamiltonian does not control the dynamics (η only in L^2_x).

One can define $G(\eta)$ for rough domains.

Arendt and ter Elst : for bounded connected open set $\Omega \subset \mathbb{R}^n$ whose boundary has a finite $(n-1)$-dimensional Hausdorff measure.

The Lipschitz threshold of regularity

One can define $G(\eta)$ for rough domains.

Arendt and ter Elst : for bounded connected open set $\Omega \subset \mathbb{R}^n$ whose boundary has a finite $(n-1)$-dimensional Hausdorff measure.

The Lipschitz threshold of regularity

Let $\eta \in W^{1,\infty}(\mathbb{R}^d)$ and $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$. There is a unique variational solution

$$\phi \in L^2\big(\mathrm{d}x\,\mathrm{d}y/(1+|y|)^2\big) \quad , \quad \nabla_{x,y}\phi \in L^2(\Omega)$$

to

$$\Delta_{x,y}\phi = 0 \quad \text{in } \{(x,y) \in \mathbb{R}^d \times \mathbb{R} \,:\, y < \eta(x)\}, \qquad \phi\big|_{y=\eta} = \psi.$$

Since ϕ is harmonic, one can define

$$G(\eta)\psi = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi\big|_{y=\eta} \in H^{-\frac{1}{2}}(\mathbb{R}^d).$$

There holds

$$\|G(\eta)\|_{H^{\frac{1}{2}}\to H^{-\frac{1}{2}}} \le C\big(\,\|\nabla\eta\|_{L^\infty}\,\big).$$

Also [Rellich,Jerison-Kenig]

$$\|G(\eta)\|_{H^1\to L^2} \le C\big(\,\|\nabla\eta\|_{L^\infty}\,\big).$$

Proposition (Agrawal - A)

For any $d \geq 1$,

$$\int_{\mathbb{R}^d} (\partial_n \phi|_{y=\eta})^2 \, \mathrm{d}x \leq 4 \int_{\mathbb{R}^d} |\nabla \psi|^2 \, \mathrm{d}x.$$

In particular

$$\|G(\eta)\|_{H^1 \to L^2} \leq 4 + 4 \|\nabla \eta\|_{L^\infty}.$$

Proof. Noether's theorem (Hamiltonian problem + invariances) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \eta(t,x) \, \mathrm{d}x = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} \int \psi(t,x) \, \mathrm{d}x = 0.$$

Remembering that

$$\partial_t \psi + g\eta + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{\big(G(\eta)\psi + \nabla \eta \cdot \nabla \psi\big)^2}{1 + |\nabla \eta|^2} = 0$$

we get

$$\int \frac{\big(G(\eta)\psi + \nabla \eta \cdot \nabla \psi\big)^2}{1 + |\nabla \eta|^2} \, \mathrm{d}x = \int |\nabla \psi|^2 \, \mathrm{d}x.$$

Remark $i)$ The rigorous proof uses the multiplier method.
$ii)$ No periodic or quasi-periodic in time solutions in finite depth.

Proposition (A - Nguyen)

Let $d \geq 1$, $\eta \in W^{1,\infty}(\mathbb{R}^d)$ *and* $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$. *There exists* c *such that*

$$\int_{\mathbb{R}^d} \psi G(\eta)\psi \, \mathrm{d}x \geq \frac{c}{1 + \|\nabla\eta\|_{\mathrm{BMO}}} \|\psi\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Remark: Let $u \in H^1(\Omega)$ and set $\psi = u|_{y=\eta}$. Denote by ϕ the harmonic extension of ψ. Then

$$\begin{aligned}\iint_\Omega |\nabla_{x,y} u|^2 \, \mathrm{d}y \, \mathrm{d}x \geq \iint_\Omega |\nabla_{x,y}\phi|^2 \, \mathrm{d}y \, \mathrm{d}x &= \int_{\partial\Omega} \phi \partial_n \phi \, \mathrm{d}\sigma \\ &= \int_{\mathbb{R}^d} \psi G(\eta)\psi \, \mathrm{d}x.\end{aligned}$$

This gives the trace inequality

$$\iint_\Omega |\nabla_{x,y} u|^2 \, \mathrm{d}y \, \mathrm{d}x \geq \frac{c}{1 + \|\nabla\eta\|_{\mathrm{BMO}}} \|u|_{y=\eta}\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Proposition (A - Nguyen)

Let $d \geq 1$, $\eta \in W^{1,\infty}(\mathbb{R}^d)$ *and* $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$. *There exists* c *such that*

$$\int_{\mathbb{R}^d} \psi G(\eta)\psi \, \mathrm{d}x \geq \frac{c}{1 + \|\nabla\eta\|_{\mathrm{BMO}}} \|\psi\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Remark: The dependence in $\|\nabla\eta\|_{\mathrm{BMO}}$ is optimal:

$$\int \psi G(\eta)\psi \, \mathrm{d}x \geq \frac{c}{(1 + \|\nabla\eta\|_{\mathrm{BMO}})^m} \|\psi\|^2_{\dot{H}^{\frac{1}{2}}} \quad \Rightarrow \quad m \geq 1.$$

Indeed, let θ be the harmonic extension of η:

$$\Delta_{x,y}\theta = 0 \quad \text{in } \{y < \eta(x)\} \quad , \quad \theta(x, \eta(x)) = \eta(x).$$

Then [Haziot-Pausader, A-Zuily]

$$0 \leq \int_{\mathbb{T}^d} \eta G(\eta)\eta \, \mathrm{d}x = \iint_{\Omega} |\nabla_{x,y}\theta|^2 \, \mathrm{d}y \, \mathrm{d}x \leq \|\eta\|_{L^\infty} \left|\mathbb{T}^d\right|.$$

Then apply the inequality with $\eta = \psi = \cos(kx)$.

Proposition (A - Nguyen)

Let $d \geq 1$ *,* $\eta \in W^{1,\infty}(\mathbb{R}^d)$ *and* $\psi \in H^{\frac{1}{2}}(\mathbb{R}^d)$ *. There exists* c *such that*

$$\int_{\mathbb{R}^d} \psi G(\eta)\psi \, \mathrm{d}x \geq \frac{c}{1 + \|\nabla \eta\|_{\mathrm{BMO}}} \|\psi\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Proof. Assume $d = 1$. Set $v(x,z) = \phi(x, z + \eta(x))$.

$$\int_{\mathbb{R}} \psi G(\eta)\psi \, \mathrm{d}x = \iint_{\Omega} |\nabla_{x,y}\phi|^2 \, \mathrm{d}y \, \mathrm{d}x = \iint_{\mathbb{R}^2_-} \left[(\partial_x v - \partial_z v \partial_x \eta)^2 + (\partial_z v)^2 \right] \mathrm{d}z \, \mathrm{d}x$$

$$\int_{\mathbb{R}} \psi \, |D_x| \, \psi \, \mathrm{d}x = \iint_{\mathbb{R}^2_-} \partial_z (v \, |D_x| \, v) \, \mathrm{d}z \, \mathrm{d}x = 2 \iint_{\mathbb{R}^2_-} (\partial_z v) \, |D_x| \, v \, \mathrm{d}z \, \mathrm{d}x$$

$$= 2 \iint_{\mathbb{R}^2_-} \left[(\partial_z v)\mathcal{H}\big(\partial_x v - (\partial_z v)\partial_x \eta\big) + (\partial_z v)\mathcal{H}\big((\partial_z v)\partial_x \eta\big) \right] \mathrm{d}z \, \mathrm{d}x.$$

Since $\mathcal{H}^* = -\mathcal{H}$, we have

$$2 \iint_{\mathbb{R}^2_-} (\partial_z v)\mathcal{H}\big((\partial_z v)\partial_x \eta\big) \, \mathrm{d}z \, \mathrm{d}x = \iint_{\mathbb{R}^2_-} (\partial_z v) \left[\mathcal{H}, \partial_x \eta \right] \partial_z v \, \mathrm{d}z \, \mathrm{d}x.$$

Apply [Coifman-Rochberg-Weiss] (see also [Lenzmann-Schikorra]). □

We have seen

$$\|G(\eta)\|_{H^{\frac{1}{2}} \to H^{-\frac{1}{2}}} + \|G(\eta)\|_{H^1 \to L^2} \le C\big(\|\nabla \eta\|_{L^\infty} \big)$$

Many other results [Nalimov, Craig-Schanz-Sulem, Wu, Beyer-Günther, Lannes]

Proposition (A.-Burq-Zuily)

(i) *For all* $s > 1 + d/2$ *and* $1/2 \le \sigma \le s$

$$\|G(\eta)\psi\|_{H^{\sigma-1}} \le C\left(\|\eta\|_{H^s}\right) \|\psi\|_{H^\sigma}.$$

(ii) *For all* $s > 1 + d/2$,

$$\|[G(\eta_1) - G(\eta_2)]\,\psi\|_{H^{s-\frac{3}{2}}} \le C\left(\|(\eta_1, \eta_2)\|_{H^{s+\frac{1}{2}}}\right) \|\psi\|_{H^s} \|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}.$$

Sobolev embedding: (i) η is Lipschitz, and (ii) $\eta_1 - \eta_2$ is not Lipschitz.

Schauder's estimates : write

$$\mathrm{div}_{x,z}(\underbrace{A(x_0, z_0)}_{\text{constant}} \nabla_{x,z} v) = \mathrm{div}_{x,z}(\underbrace{(A(x_0, z_0) - A(x, z))}_{\text{small}} \nabla_{x,z} v).$$

To prove Schauder's estimates, it is convenient to paralinearize.

Paralinearization

By using the Fourier transform : $G(0) = |D_x|$.

If $\eta \in C^\infty$, it is known since Calderón that $G(\eta)$ is a Ψ DO of order 1, whose principal symbol is

$$\lambda(x,\xi) := \sqrt{(1 + |\nabla\eta(x)|^2)\,|\xi|^2 - (\nabla\eta(x)\cdot\xi)^2}.$$

More precisely,

$$G(\eta)\psi = (2\pi)^{-d}\int e^{ix\cdot\xi}\lambda(x,\xi)\widehat{\psi}(\xi)\,d\xi \;+\; R_0(\eta)f,$$

where the remainder is of order 0, satisfying [Lannes]

$$\exists K \geq 1, \forall s \geq \frac{1}{2}, \quad \|R_0(\eta)\psi\|_{H^s} \leq C\left(\|\eta\|_{H^{s+K}}\right)\|\psi\|_{H^s}.$$

Remarks: $i)$ λ well-defined for any $\eta \in W^{1,\infty}(\mathbb{R}^d)$.

$ii)$ If $d = 1$ or $\eta = 0$ then $\lambda(x,\xi) = |\xi|$ and $\mathrm{Op}(\lambda) = |D_x|$.

$iii)$ K corresponds to a loss of derivatives (Lannes, Iguchi).

Paraproducts:

$$\begin{aligned} au &= \frac{1}{(2\pi)^{2d}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{ix\cdot(\xi_1+\xi_2)}\, \widehat{a}(\xi_1)\, \widehat{u}(\xi_2)\, d\xi_1 d\xi_2 \\ &= \iint_{|\xi_1+\xi_2|\sim|\xi_2|} + \iint_{|\xi_1+\xi_2|\sim|\xi_1|} + \iint_{|\xi_1|\sim|\xi_2|} \\ &= T_a u + T_u a + R(a,u) \end{aligned}$$

$T_a u$ has the same regularity as u for $a \in L^\infty$, and $R(a,u)$ is twice more regular.

Theorem (Bony, paralinearization of a product).

$$\begin{array}{lllll} \forall \sigma \in \mathbb{R} & a \in L^\infty(\mathbb{R}^d) & u \in H^\sigma(\mathbb{R}^d) & \Rightarrow & T_a u \in H^\sigma(\mathbb{R}^d) \\ \forall s > 0 & a \in H^s(\mathbb{R}^d) & u \in H^s(\mathbb{R}^d) & \Rightarrow & R(a,u) \in H^{2s-\frac{d}{2}}(\mathbb{R}^d) \end{array}$$

In dimension one, this simplifies to

$$G(\eta)\psi = |D_x|\,\psi \;+\; R(\eta)\psi,$$

where $R(\eta)$ is a smoothing operator,

Question: compute $R(\eta) = G(\eta)\psi - |D_x|\,\psi$.

$* * *$ First computation $* * *$

Theorem (A-Métivier). Let $3 < \gamma < s$. Set $V = (\partial_x \phi)|_{y=\eta}$ and $B = (\partial_y \phi)|_{y=\eta}$. Then

$$G(\eta)\psi = |D_x|\,(\psi - T_B\eta) - \partial_x\big(T_V\eta\big) + F$$

where

$$\|F\|_{H^{s+\gamma-4}} \le C\left(\|\eta\|_{C^\gamma}\right)\left\{\|\psi\|_{C^\gamma}\,\|\eta\|_{H^s} + \|\eta\|_{C^\gamma}\,\big\|\psi\big\|_{H^s}\right\}.$$

Extensions: ABZ, Thibault de Poyferré, Albert Ai, Xuecheng Wang, Fan Zheng, Chenyang Zhou...

Step 1 : Paracomposition. We flatten the boundary via the diffeomorphism

$$\chi\colon (x,z)\mapsto (x,z+\eta(x)).$$

Set $v(x,z)=\phi(x,z+\eta(x))$. Then by elliptic regularity $v\in H^{s+\frac12}\big(\mathbb{R}^d\times[-1,0]\big)$.

By paralinearization, we get

$$T_{1+|\nabla\eta|^2}\partial_z^2 v+\Delta v-2T_{\nabla\eta}\cdot\nabla\partial_z v-T_{\Delta\eta}\partial_z v\in C^0_z\big([-1,0];H^{s-2}_x(\mathbb{R}^d)\big)$$

By using Alinhac's paracomposition operators, introduce

$$u=\phi\circ\chi-T_{\phi'\circ\chi}\chi=v-T_{\partial_z v}\eta.$$

Then u satisfies a paradifferential elliptic equation:

$$T_{1+|\nabla\eta|^2}\partial_z^2 u+\Delta u-2T_{\nabla\eta}\cdot\nabla\partial_z u-T_{\Delta\eta}\partial_z u\in C^0_z\big([-1,0];H^{2s-\frac{5+d}{2}}_x(\mathbb{R}^d)\big)$$

We call u the good unknown of Alinhac.

Step 2 : elliptic factorization. There exist two symbols a, A such that

$$(\partial_z - T_a)(\partial_z - T_A)u \in C^0_z\big([-1,0]; H^{2s-K(d)}_x(\mathbb{R}^d)\big).$$

Step 3: elliptic regularity. Introduce $w := (\partial_z - T_A)u$, then

$$\partial_z w - T_a w \sim 0.$$

and hence

$$(\partial_z u - T_A u)|_{z=0} = w(0) \sim 0.$$

This gives $\partial_z u$ **on the boundary** $\{z=0\}$ **in terms of tangential derivatives** + a smooth remainder. □

Many applications to the water-wave problem

- The Cauchy problem (cf lectures by Tataru and Wu)
- Balanced energy estimates: *Ai-Ifrim-Tataru*
- Enhanced existence :

  Shatah, Delort-Szeftel, Hunther-Ifrim, Wu, Germain-Masmoudi-Shatah, A-Delort, Ionescu-Pusateri, Hunther-Ifrim-Tataru, Wang, Berti-Feola-Franzoi, Deng-Ionescu-Pusateri, Ehrnström-Wang

- Qualitative properties of the flow map : *de Poyferré, Said*
- Dispersive estimates : see later
- Small divisors : *Iooss-Plotnikov-Toland, A-Baldi, Berti-Montalto*
- Control theory : *A-Baldi-Han Kwan, Zhu*

Theorem (Matioc ; A-Meunier-Smets ; Nguyen-Pausader). The Cauchy problem for the Hele-Shaw equation $\partial_t \eta + G(\eta)\eta = 0$ is LWP on $H^s(\mathbb{T}^d)$ for $s > 1 + d/2$.

Proof. One quasilinearizes (HS) as follows.

Lemma 1. Let $\phi = \phi(t,x,y)$ harmonic extension of η in $\{y < \eta(t,x)\}$ and

$$a = 1 - (\partial_y \phi)|_{y=\eta} \quad , \quad V = -(\nabla_x \phi)|_{y=\eta}.$$

Then

$$\partial_t V + V \cdot \nabla V + aG(\eta)V + \frac{\gamma}{a} V = 0 \quad \text{where}$$

$$\gamma = \frac{1}{1+|\nabla\eta|^2}\Big(G(\eta)(a^2+V^2) - 2aG(\eta)a - 2V \cdot G(\eta)V\Big).$$

Proof: shape derivative formula [Zakharov, Lannes]

$$\partial G(\eta)\psi = -G(\eta)(\partial\psi - \mathfrak{B}\partial\eta) - \operatorname{div}(\mathfrak{V}\partial\eta)$$

$$\mathfrak{B} = \frac{\nabla\eta \cdot \nabla\psi + G(\eta)\psi}{1+|\nabla\eta|^2}, \quad \mathfrak{V} = \nabla\psi - \mathfrak{B}\nabla\eta.$$

Theorem (Matioc ; A-Meunier-Smets ; Nguyen-Pausader). The Cauchy problem for the Hele-Shaw equation $\partial_t \eta + G(\eta)\eta = 0$ is LWP on $H^s(\mathbb{T}^d)$ for $s > 1 + d/2$.

Proof. One quasilinearizes (HS) as follows.

Lemma 1. Let $\phi = \phi(t, x, y)$ harmonic extension of η in $\{y < \eta(t, x)\}$ and

$$a = 1 - (\partial_y \phi)|_{y=\eta} \quad , \quad V = -(\nabla_x \phi)|_{y=\eta}.$$

Then

$$\partial_t V + V \cdot \nabla V + aG(\eta)V + \frac{\gamma}{a} V = 0 \quad \text{where}$$

$$\gamma = \frac{1}{1 + |\nabla \eta|^2} \Big(G(\eta)(a^2 + V^2) - 2aG(\eta)a - 2V \cdot G(\eta)V \Big).$$

Lemma 2. $a > 0$ [Wu for WW]

Proof: the function $y - \phi$ is harmonic and vanishes on the boundary. The Hopf-Zaremba principle gives $\partial_n (y - \phi) > 0$. $\square$

Recall that

$$\partial_t V + V \cdot \nabla V + aG(\eta)V + \frac{\gamma}{a} V = 0$$

Notice that $A = V \cdot \nabla$ is of order 1 but

$$(A + A^*)f = -(\operatorname{div} V)f \quad \text{is of order } 0 .$$

Bounded from L^2 to L^2 if V is $W_x^{1,\infty}$.
Loosely speaking, $A + A^*$ is of order $1 - \varepsilon$ provided that V is $L_t^\infty(C^{0,\varepsilon})$.

Recall that

$$\partial_t V + V \cdot \nabla V + aG(\eta)V + \frac{\gamma}{a} V = 0$$

Notice that $A = V \cdot \nabla$ is of order 1 but

$$(A + A^*)f = -(\operatorname{div} V)f \quad \text{is of order } 0 .$$

Bounded from L^2 to L^2 if V is $W_x^{1,\infty}$.
Loosely speaking, $A + A^*$ is of order $1 - \varepsilon$ provided that V is $L_t^\infty(C^{0,\varepsilon})$.

Last step : paradifferential analysis of $G(\eta)$ [A.-Burq-Zuily]

Lemma 3. Assume that $\eta \in H^s(\mathbb{R}^d)$ with $s = 1 + \frac{d}{2} + \varepsilon$, $\varepsilon > 0$.

If $d = 1$ then $G(\eta) = |D_x| + R(\eta)$ where $|D_x| = \sqrt{-\partial_{xx}}$ and

$$\|R(\eta)\|_{H^\mu \to H^{\mu-1+\varepsilon}} \leq C(\|\eta\|_{H^s}) \quad \text{for} \quad \frac{1}{2} \leq \mu \leq s - \frac{1}{2}.$$

If $d \geq 2$ same result with paradifferential operators.

Then energy estimates and interpolation arguments. □

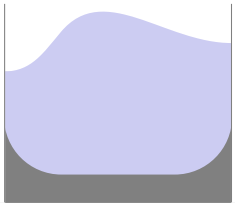


Figure: 2D section of the channel

Application : the Cauchy problem in a channel

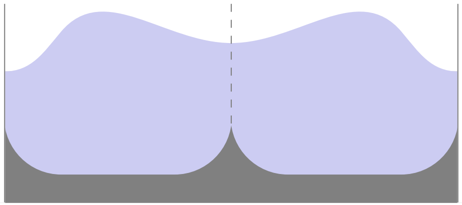


Figure: 2D section of the channel

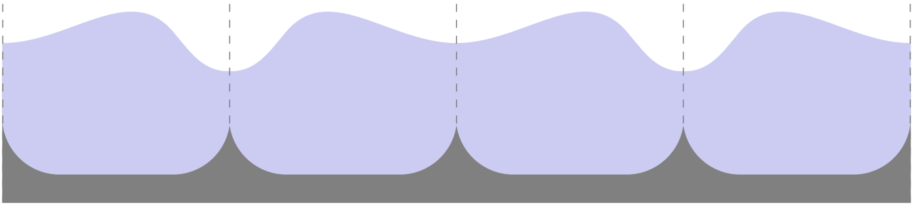


Figure: 2D section of the channel

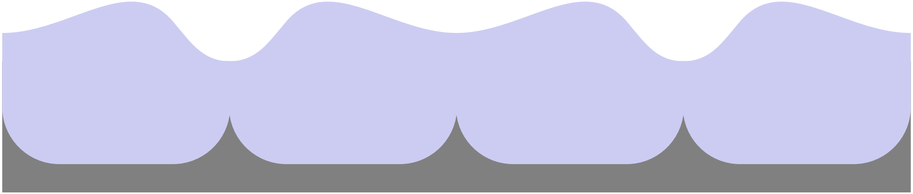


Figure: 2D section of the channel

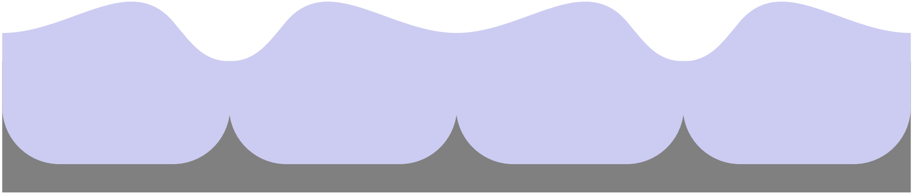


Figure: 2D section of the channel

For $s < 3$, if $\partial_{x_1}\eta(0,x_2) = 0 = \partial_{x_1}\eta(1,x_2)$ then

$$\eta \in H^{s+\frac{1}{2}}\big((0,1)_{x_1} \times \mathbb{R}_{x_2}\big) \;\Rightarrow\; \underline{\eta} \in H^{s+\frac{1}{2}}\big(\mathbb{T} \times \mathbb{R}_{x_2}\big).$$

- ABZ: LWP for $s > 3$ with surface tension and $3D$ fluids.
- Semi-classical Strichartz estimates (Lebeau, Smith, Tataru, Bahouri-Chemin, Staffilani-Tataru, Burq-Gérard-Tzvetkov):

Christianson-Hur-Staffilani, ABZ, de Poyferré, Ai

Application : reduction to constant coefficients in $1D$

(Oversymplifying) One can rewrite the WW system as [A-Burq-Zuily]

$$Pu = \frac{\partial u}{\partial t} + V(u)\partial_x u + i\,|D_x|^{\frac{3}{4}}\left(c(u)\,|D_x|^{\frac{3}{4}}\,u\right) = 0$$

where $x \in \mathbb{T}$ and

$$V(u) = \mathrm{Re}\left(\langle D_x\rangle^{-N} u\right)$$

with N as large as we want (for smooth enough initial data).

Using a change of variables (preserving the L^2-norm in x)

$$h(t,x) \mapsto (1 + \partial_x \kappa(t,x))^{\frac{1}{2}}\, h(t, x + \kappa(t,x))$$

we replace P by

$$Q = \partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R, \quad R \text{ is of order zero}$$

where one can further assume that $\int_{\mathbb{T}} W(t,x)\,dx = 0$.

To study $\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}} + R'$, we seek an operator A such that

$$\left[A, i\,|D_x|^{\frac{3}{2}}\right] + W\partial_x A \quad \text{is a zero order operator}$$

We find [A.-Baldi]

$$A = \mathrm{Op}\left(q(t,x,\xi)e^{i\beta(t,x)|\xi|^{\frac{1}{2}}}\right)$$

with

$$\beta = \beta_0(t) + \frac{2}{3}\partial_x^{-1}W.$$

Then

$$\left(\partial_t + W\partial_x + i\,|D_x|^{\frac{3}{2}}\right)A = A\left(\partial_t + i\,|D_x|^{\frac{3}{2}} + R''\right)$$

with R'' of order 0.

Notice that $A \in \mathrm{Op}\,S^0_{\rho,\rho}$ with $\rho = 1/2$ ([Said 2020], quasi-linear).

For Benjamin-Ono, similar conjugation with $A \in \mathrm{Op}\,S^0_{1,0}$ (semi-linear).

The Muskat problem

Dynamics of the curve $\Sigma(t)$ separating two fluids:

$$\Omega_1(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}\,;\, y > f(t,x)\}$$

$$\Omega_2(t) = \{(x,y) \in \mathbb{R} \times \mathbb{R}\,;\, y < f(t,x)\}$$

$$\Sigma(t) = \{y = f(t,x)\}.$$

Each Ω_i is occupied by an incompressible fluid with constant density ρ_i, velocity u_i and pressure P_i. Muskat equations read:

$u_i = -\nabla_{x,y}(P_i + \rho_i g y)$	in Ω_i	(Darcy)
$\operatorname{div}_{x,y} u_i = 0$	in Ω_i	(incompressibility)
$P_1 = P_2$	on Σ	(continuity of P)
$u_1 \cdot n = u_2 \cdot n = \dfrac{\partial_t f}{\sqrt{1 + (\partial_x f)^2}}$	on Σ	(transport of Σ)

One can reduce the Muskat problem to a parabolic evolution equation for f:

Caflisch-Orellana-Siegel, Escher-Simonett, Ambrose

Córdoba and Gancedo obtained a beautiful compact formulation:

$$\partial_t f = \frac{\rho}{2\pi}\partial_x \int_{\mathbb{R}} \arctan(\Delta_\alpha f)\,\mathrm{d}\alpha \qquad \Delta_\alpha f(t,x) = \frac{f(t,x)-f(t,x-\alpha)}{\alpha}$$

where $\rho = \rho_2 - \rho_1$. We assume $\rho > 0$ and set $\rho = 2$.

Scaling $f(t,x) \mapsto \frac{1}{\lambda} f(\lambda t, \lambda x) \longrightarrow$ Critical spaces $\dot{H}^{\frac{3}{2}}(\mathbb{R}),\ \dot{W}^{1,\infty}(\mathbb{R})$.

- Many proofs of well-posed on sub-critical spaces

  Yi, Caflisch-Howison-Siegel, Ambrose, Córdoba-Gancedo, Córdoba-Córdoba-Gancedo, Cheng- Granero-Belinchón- Shkoller, Constantin-Gancedo-Shvydkoy-Vicol, Matioc, Deng-Lei-Lin, A-Lazar, Nguyen-Pausader

Cameron first studied well-posedness for interfaces in $\dot{W}^{1,\infty} \cap L^2(\mathbb{R})$ for initial data whose slope is bounded by 1.

Since $|D_x| = \partial_x \mathcal{H}$ with

$$\mathcal{H}f(x) = \frac{1}{\pi}\mathrm{pv}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y$$

we have

$$|D_x|\, f = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{R}} \partial_x \Delta_\alpha f \,\mathrm{d}\alpha$$

Then write

$$\partial_x \arctan(\Delta_\alpha f) = \frac{\partial_x \Delta_\alpha f}{1+(\Delta_\alpha f)^2} = \partial_x \Delta_\alpha f - (\partial_x \Delta_\alpha f)\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}$$

to get

$$\partial_t f + |D_x|\, f = \mathcal{T}(f)f \quad \text{where} \quad \mathcal{T}(f)f = -\frac{1}{\pi}\int_{\mathbb{R}} (\partial_x \Delta_\alpha f)\frac{(\Delta_\alpha f)^2}{1+(\Delta_\alpha f)^2}\,\mathrm{d}\alpha.$$

This gives a rigorous meaning to the Muskat equation (A.-Lazar).

Prop. The map $f \mapsto \mathcal{T}(f)f$ is locally Lipschitz from $H^{\frac{3}{2}}(\mathbb{R})$ to $L^2(\mathbb{R})$.

$$\|uv\|_{L^2} \le \|u\|_{L^4}\,\|u\|_{L^4}\,, \quad H^{\frac{1}{2}} \subset L^4, \quad \|u\|^2_{\dot{H}^{\frac{1}{2}}} \sim \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|}\,\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|}.$$

Theorem (Córdoba and Lazar). GWP for $f_0 \in H^{\frac{5}{2}}(\mathbb{R})$ such that

$$\left(1 + \|\partial_x f_0\|_{L^\infty}^4\right) \|f_0\|_{\dot{H}^{\frac{3}{2}}} \ll 1.$$

$i)$ Main estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \,\mathrm{d}x \lesssim \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 \right) \|f\|_{\dot{H}^2}^2$$

Very delicate estimate since $H^{\frac{1}{2}}(\mathbb{R})$ is not a Banach algebra.

Proof based on a *reformulation* in terms of oscillatory integrals, *Besov* spaces.

See also Gancedo-Lazar, Granero-Bellinchón & Scrobogna, Scrobogna.

Theorem (Córdoba and Lazar). GWP for $f_0 \in H^{\frac{5}{2}}(\mathbb{R})$ such that

$$(1 + \|\partial_x f_0\|_{L^\infty}^4) \|f_0\|_{\dot{H}^{\frac{3}{2}}} \ll 1.$$

$i)$ Main estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \,\mathrm{d}x \lesssim \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 \right) \|f\|_{\dot{H}^2}^2$$

Very delicate estimate since $H^{\frac{1}{2}}(\mathbb{R})$ is not a Banach algebra.

Proof based on a *reformulation* in terms of oscillatory integrals, *Besov* spaces.

See also Gancedo-Lazar, Granero-Bellinchón & Scrobogna, Scrobogna.

Theorem (Córdoba and Lazar). GWP for $f_0 \in H^{\frac{5}{2}}(\mathbb{R})$ such that

$$(1 + \|\partial_x f_0\|_{L^\infty}^4) \|f_0\|_{\dot{H}^{\frac{3}{2}}} \ll 1.$$

$i)$ Main estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + \int_{\mathbb{R}} \frac{(\partial_{xx} f)^2}{1 + (\partial_x f)^2} \, \mathrm{d}x \lesssim \left(\|f\|_{\dot{H}^{\frac{3}{2}}} + \|f\|_{\dot{H}^{\frac{3}{2}}}^2 \right) \|f\|_{\dot{H}^2}^2$$

Very delicate estimate since $H^{\frac{1}{2}}(\mathbb{R})$ is not a Banach algebra.

Proof based on a *reformulation* in terms of oscillatory integrals, *Besov* spaces.

See also Gancedo-Lazar, Granero-Bellinchón & Scrobogna, Scrobogna.

$ii)$ Bound for the slope: to control the denominator $1 + (\partial_x f)^2$. By an integration by parts argument (C.G., Cameron, G.L.):

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x f\|_{L^\infty} \lesssim \|f\|_{H^2}^2$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f\|_{\dot{H}^{\frac{3}{2}}}^2 + c_0 \int_{\mathbb{R}} (\partial_{xx} f)^2 \, \mathrm{d}x \leq 0.$$

Theorem (A & Q.-H Nguyen). GWP for $f_0 \in H^{\frac{3}{2}}(\mathbb{R})$ such that $\|f_0\|_{\dot{H}^{\frac{3}{2}}} \ll 1.$

A null-type property.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\dot{H}^{\frac{3}{2}}} + \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1+(\partial_x f)^2}\,\mathrm{d}x \lesssim \left(1+\|f\|^7_{\dot{H}^{\frac{3}{2}}}\right)\|f\|_{\dot{H}^{\frac{3}{2}}} \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1+(\partial_x f)^2}\,\mathrm{d}x$$

Theorem (A & Q.-H Nguyen). GWP for $f_0 \in H^{\frac{3}{2}}(\mathbb{R})$ such that $\|f_0\|_{\dot{H}^{\frac{3}{2}}} \ll 1.$

A null-type property.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\dot{H}^{\frac{3}{2}}} + \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1+(\partial_x f)^2}\,\mathrm{d}x \lesssim \left(1+\|f\|^7_{\dot{H}^{\frac{3}{2}}}\right)\|f\|_{\dot{H}^{\frac{3}{2}}} \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1+(\partial_x f)^2}\,\mathrm{d}x$$

Imply at once GWP for small data: if $\|f\|_{\dot{H}^{\frac{3}{2}}}$ is small enough then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\dot{H}^{\frac{3}{2}}} + c_0 \int_{\mathbb{R}} \frac{(\partial_{xx}f)^2}{1+(\partial_x f)^2}\,\mathrm{d}x \le 0.$$

Theorem (A & Q.-H Nguyen). LWP in the critical space $H^{\frac{3}{2}}(\mathbb{R})$.

A norm depending on the initial data: for $f_0 \in H^{\frac{3}{2}}(\mathbb{R})$ there is a weight κ_0 s.t.

$$\|f_0\|^2_{\mathcal{H}^{\frac{3}{2},\kappa_0}} := \int_{\mathbb{R}} (1+|\xi|)^3 \kappa_0(|\xi|)^2 \left|\hat{f}_0(\xi)\right|^2 \mathrm{d}\xi < +\infty \quad \text{and} \quad \lim_{|\xi|\to+\infty} \kappa_0(|\xi|) = +\infty.$$

We estimate the $L^\infty_t(\mathcal{H}^{\frac{3}{2},\kappa_0})$-norm of f.

Muskat equation reads

$$\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \, \mathrm{d}\alpha$$

One expects to extract a formulation of the form

$$\partial_t f + V \partial_x f + \gamma \left| D_x \right| f = R(f)$$

for some coefficients V and γ depending on $\partial_x f$.

$\longrightarrow$ paradifferential type analysis ; A.-Lazar and Nguyen-Pausader.

The approach in [Nguyen-Pausader] applies for many physical equations.

The approach in [A.-Lazar] is adapted to study critical problem.

Following Shnirelman, we consider a simpler version of paraproducts:

$$\tilde{T}_a g = \Lambda^{-(1+\varepsilon)}(a \Lambda^{1+\varepsilon} g) \quad , \quad \Lambda = (I - \partial_{xx})^{1/2}$$

and we decompose

$$\mathcal{O}\left(\alpha, \cdot\right) = \frac{1}{2}\left(\frac{1}{1 + (\Delta_\alpha f)^2} - \frac{1}{1 + (\Delta_{-\alpha} f)^2} \right)$$
$$\mathcal{E}\left(\alpha, \cdot\right) = \frac{1}{2}\left(\frac{1}{1 + (\Delta_\alpha f)^2} + \frac{1}{1 + (\Delta_{-\alpha} f)^2} \right)$$

Since $\Delta_\alpha f(x) \to f_x(x)$ when $\alpha \to 0$, decompose

$$\mathcal{E}(\alpha, \cdot) = \frac{1}{1+(\partial_x f)^2} + \left(\mathcal{E}(\alpha, \cdot) - \frac{1}{1+(\partial_x f)^2}\right).$$

By considering that $|D_x| f = -\frac{1}{\pi}\int_{\mathbb{R}} \partial_x \Delta_\alpha f \, \mathrm{d}\alpha$ we obtain

$$\partial_t f + \frac{1}{1+(\partial_x f)^2} |D_x| f + V(f)\partial_x f + R = 0 \qquad (\star\star)$$

with

$$V = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\mathcal{O}(\alpha, .)}{\alpha} \, \mathrm{d}\alpha, \qquad R = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{f_x(\cdot - \alpha)}{\alpha}\left(\frac{1}{1+(\Delta_\alpha f)^2} - \frac{1}{1+f_x^2}\right) \mathrm{d}\alpha.$$

Then, we estimate the $L^\infty_t(\mathcal{H}^{\frac{3}{2},\kappa}(\mathbb{R}))$-norm of f by

- commuting $|D_x| \kappa(D_x)$ with the equation $(\star\star)$
- taking the L^2_x scalar product with $|D_x|^2 \kappa(D_x) f$ and integrating in time.

Thank you!