Stratospheric planetary flows from the perspective of the Euler equation on a rotating sphere

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The stratospheric dynamics on the giant planets of our solar system is dominated by zonal flows that feature a banded structure, with superimposed non-zonal patterns, in the form of long-lived vortices ("eddies") – typically small but sometimes large (like Jupiter's Great Red Spot and Saturn's Polar Hexagon). Since the stratosphere is thermally stably stratified and viscous effects are not relevant, two-dimensional inviscid flows on a rotating sphere are pertinent models.

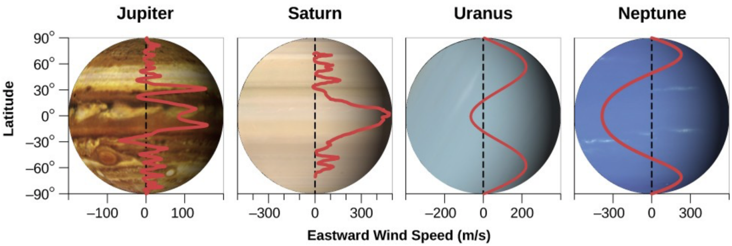

Figure: Variation of the mean zonal winds with latitude on the giant planets of our solar system, measured relative to the planet's rotation speed about its polar axis (Credit: OpenStax CNX). The traces of methane (which absorbs red light) in their upper atmosphere gives Uranus and Neptune a blue hue, obscuring the visibility of specific flow patterns. These pictures show the high altitude clouds just beneath the stratosphere (at the top of the troposphere) – the only planetary atmosphere in our solar system transparent enough to see through from space being that of the Earth.

Differential geometry of the sphere

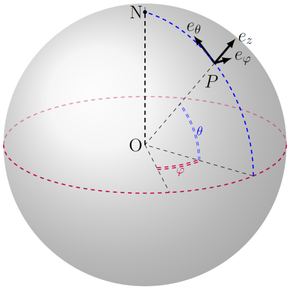

Figure: The rotating spherical coordinate system (r', φ, θ): $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the angle of latitude, $\varphi \in [-\pi, \pi]$ is the angle of longitude, and $r' = |OP|$ is the distance from the origin at planet's center. The North Pole is at $\theta = \frac{\pi}{2}$, the Equator is on $\theta = 0$ and the South Pole is at $\theta = -\frac{\pi}{2}$. The double-valued ambiguity along the international date line $\varphi = \pi$ can be resolved by assuming a periodic dependence on the azimuthal angle φ but the unit vectors $\mathbf{e}_\varphi$ and $\mathbf{e}_\theta$ are not well-defined at the poles, where the latitude circles degenerate into a single point. Note that, by the "hairy ball theorem", the 2-sphere $\mathbb{S}^2$ does not possess a continuously differentiable field of unit tangent vectors – it is not possible to cover $\mathbb{S}^2$ with one chart.

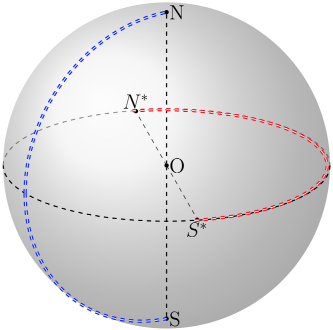

Figure: The longitude-latitude spherical coordinates $(\varphi, \theta) \in (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ provides a chart $(\varphi, \theta) \in (-\pi, \pi) \times \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \mapsto (\cos\varphi \cos\theta,\ \sin\varphi \cos\theta,\ \sin\theta) \in \mathbb{S}^2$ covering $\mathbb{S}^2$ with the half-circle $\varphi = \pi$ (the international date line, including the poles) excised. A smooth atlas for $\mathbb{S}^2$ is obtained by coupling this with the chart $(\tilde{\varphi}, \tilde{\theta}) \in (-\pi, \pi) \times \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \mapsto (-\cos\tilde{\varphi}\cos\tilde{\theta},\ \sin\tilde{\theta},\ \sin\tilde{\varphi}\cos\tilde{\theta}) \in \mathbb{S}^2$ covering $\mathbb{S}^2$ with the equatorial half-circle parametrized in spherical coordinates by $\{\theta = 0\,,\ \varphi \in [0, \pi]\}$ excised: the bijective transformation $(x, y, z) \mapsto (-x, z, y)$ between the above parametrizations is equivalent to first rotating the Euclidean coordinate system by $\frac{\pi}{2}$ about the x-axis and then by π about the z-axis.

The 4-dimensional tangent bundle $T\mathbb{S}^2$ of the 2-sphere $\mathbb{S}^2$ is not parallelizable as a consequence of the hairy ball theorem. However, at every point X of $\mathbb{S}^2 \setminus \{N, S\}$, having spherical coordinates $(\varphi, \theta) \in [-\pi, \pi] \times (-\frac{\pi}{2}, \frac{\pi}{2})$, the tangent vectors

$$\mathbf{e}_\varphi = \frac{1}{\cos\theta}\,\partial_\varphi, \qquad \mathbf{e}_\theta = \partial_\theta\,,$$

provide us with a basis of the tangent space $T_X\mathbb{S}^2$ at $X \in \mathbb{S}^2$. In these coordinates, the Riemannian volume element is

$$\mathrm{d}\sigma = \cos\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta\,,$$

and the classical differential operators (gradient and Laplace-Beltrami for scalar functions $\psi : \mathbb{S}^2 \to \mathbb{R}$, divergence for vector fields $F : \mathbb{S}^2 \to T\mathbb{S}^2$) are given by

$$\begin{aligned}
&\mathrm{grad}\,\psi = \partial_\theta\psi\,\mathbf{e}_\theta + \frac{\partial_\varphi\psi}{\cos\theta}\,\mathbf{e}_\varphi\,,\\
&\mathrm{div}(F_\varphi\,\mathbf{e}_\varphi + F_\theta\,\mathbf{e}_\theta) = \frac{1}{\cos\theta}\,[\partial_\varphi F_\varphi + \partial_\theta(\cos\theta\,F_\theta)]\,,\\
&\Delta\,\psi = \mathrm{div}\,\mathrm{grad}\,\psi = \partial_\theta^2\psi - \tan\theta\,\partial_\theta\psi + \frac{1}{(\cos\theta)^2}\,\partial_\varphi^2\psi\,.
\end{aligned}$$

The Laplace-Beltrami operator Δ on $\mathbb{S}^2$, operating in the Hilbert space $L^2(\mathbb{S}^2)$ obtained as the completion of the smooth functions $f : \mathbb{S}^2 \to \mathbb{C}$ of zero mean (i.e., with $\iint_{\mathbb{S}^2} f \, d\sigma = 0$) with respect to the inner product

$$\langle f_1, f_2 \rangle = \iint_{\mathbb{S}^2} f_1 \, \overline{f_2} \, d\sigma \,,$$

(where the overbar denotes complex conjugation) is negative, self-adjoint and its spectrum is the discrete set of eigenvalues $\bigcup_{j\geq 1}\{-j(j+1)\}$, the spherical harmonics $\{Y_j^m\}_{j\geq 1,\,|m|\leq j}$ being an orthonormal basis of eigenfunctions in $L^2(\mathbb{S}^2)$, with

$$\Delta Y_j^m = -j(j+1) Y_j^m \,, \qquad j \geq 1 \,, \; m \in \{-j, \dots, j\} \,.$$

A basis of the j-th eigenspace $\mathbb{E}_j$, of dimension $2j+1$, associated to the eigenvalue $-j(j+1)$, is provided by the $(2j+1)$ spherical harmonics $Y_j^m(\varphi, \theta)$ of degree j and zonal number m $(-j \leq m \leq j)$.

Stratospheric flow is inviscid, being governed by the system†

$$\frac{\partial u_0}{\partial t} + \frac{u_0}{\cos\theta}\frac{\partial u_0}{\partial \varphi} + v_0\frac{\partial u_0}{\partial \theta} - u_0 v_0 \tan\theta - 2\omega\, v_0 \sin\theta = -\frac{1}{\rho_0 \cos\theta}\frac{\partial p_0}{\partial \varphi}\,,$$

$$\frac{\partial v_0}{\partial t} + \frac{u_0}{\cos\theta}\frac{\partial v_0}{\partial \varphi} + v_0\frac{\partial v_0}{\partial \theta} + u_0^2 \tan\theta + 2\omega\, u_0 \sin\theta + \omega^2 \sin\theta\cos\theta = -\frac{1}{\rho_0}\frac{\partial p_0}{\partial \theta}\,,$$

$$0 = \frac{1}{\rho_0}\frac{\partial p_0}{\partial z} + g\,,$$

$$\frac{\partial u_0}{\partial \varphi} + \frac{\partial}{\partial \theta}(v_0 \cos\theta) = 0\,.$$

Using the third equation we introduce the stream function, $\psi(\varphi, \theta, z, t)$, with

$$u_0 = -\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_0 = \frac{1}{\cos\theta}\frac{\partial \psi}{\partial \varphi}\,, \tag{2}$$

while the elimination of the dynamic pressure p_0 between the first two equations gives the vorticity equation

$$\frac{\partial}{\partial t}\Delta\psi + \frac{1}{\cos\theta}\left[\frac{\partial \psi}{\partial \varphi}\frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial \varphi}\right]\left(\nabla_\Sigma^2 \psi + 2\omega\sin\theta\right) = 0\,. \tag{3}$$

† For ρ_0 constant the system (1) particularizes to that describing inviscid flow on the surface of a rotating sphere. This 2D character is related to the fact that, due to an ascending temperature with height (the equation of state being $p = \rho T$), the stratosphere is stably stratified and vertical motion is suppressed.

Stationary solutions of (3) lead to travelling-wave solutions describing stratospheric flows, with an associated decrease of density and increase of temperature with height. Indeed, given the vertical density stratification of the stratosphere $\rho_0(z)$, if $\psi_0(\varphi, \theta)$ solves

$$\Delta\psi_0 = F(\psi_0)$$

for some $F \in C^1(\mathbb{R}, \mathbb{R})$, then

$$\psi(\varphi, \theta, z, t) = \omega \sin\theta + \frac{1}{\sqrt{\rho_0(z)}}\, \psi_0(\varphi + \omega\, t, \theta)$$

with the associated pressure

$$p_0(\varphi, \theta, z, t) = \mathcal{F}(\psi_0(\varphi + \omega\, t, \theta)) - \frac{1}{2}\Big(\frac{\partial\psi_0}{\partial\theta}\,(\varphi + \omega\, t, \theta)\Big)^2 - \frac{1}{2\cos^2\theta}\Big(\frac{\partial\psi_0}{\partial\varphi}\,(\varphi + \omega\, t, \theta)\Big)^2 - g\int_0^z \rho_0(s)\,\mathrm{d}s\,,$$

where $\mathcal{F}$ is a primitive of F, is a solution of the system, describing height-dependent stratospheric planetary flows that propagate zonally westwards.

The Euler equation for inviscid flow on a rotating sphere

The Euler equation on a sphere rotating at speed ω about the polar axis can be written for either the stream function ψ, the velocity field

$$U = (u, v) \quad \text{with} \quad \begin{cases} u = -\partial_\theta \psi \,, \\ v = \frac{1}{\cos\theta}\, \partial_\varphi \psi \,. \end{cases}$$

or the vorticity $\Omega = \Delta\psi$. The use of spherical coordinates (more precisely, the fact that longitude is not well-defined and latitude circles degenerate into a single point at the poles) introduces artificial singularities at the poles that can be ruled out either by switching to the chart that covers the sphere with the equatorial half-circle removed or by taking smoothness into account. For example, regarding the apparent singularity of the meridional velocity component v at the poles (where $\cos\theta = 0$), note that for any C^1-function $\psi : \mathbb{S}^2 \to \mathbb{R}$ the continuity of the gradient with respect to the spherical coordinates (φ, θ) implies

$$\lim_{\theta \to \pm \frac{\pi}{2}} \partial_\varphi \psi(\varphi, \theta) = 0$$

since on any genuine circle of latitude $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the periodicity of ψ in the longitudinal direction ensures the existence of a point where $\partial_\varphi \psi$ vanishes.

The Euler equation set on the sphere $\mathbb{S}^2$, in a frame rotating at speed $\omega \in \mathbb{R}$ about the polar axis, can be written in terms of the stream function ψ as

$$\partial_t \Delta\psi + \frac{1}{\cos\theta}\left[-\partial_\theta\psi\,\partial_\varphi + \partial_\varphi\psi\,\partial_\theta\right](\Delta\psi + 2\omega\sin\theta) = 0\,. \qquad (\mathcal{E}_\omega)$$

To express the Euler equation in terms of the velocity field we have to add the divergence-free condition to the evolution equation $(\mathcal{E}_\omega)$, together with an auxiliary scalar pressure field p (that arises as a Lagrange multiplier for the divergence-free constraint)

$$\begin{cases} D_t U + 2\omega\sin\theta\, JU = -\operatorname{grad} p\,, \\ \operatorname{div} U = 0\,, \end{cases} \qquad (4)$$

where D_t is the material derivative, describing the transport by the velocity field U, given by

$$D_t = \partial_t + \nabla_U = \partial_t + u\nabla_{\mathbf{e}_\varphi} + v\nabla_{\mathbf{e}_\theta}\,,$$

in terms of the covariant derivatives

$$\nabla_{\mathbf{e}_\theta}\mathbf{e}_\theta = \nabla_{\mathbf{e}_\theta}\mathbf{e}_\varphi = 0\,, \qquad \nabla_{\mathbf{e}_\varphi}\mathbf{e}_\theta = -\tan\theta\,\mathbf{e}_\varphi\,, \qquad \nabla_{\mathbf{e}_\varphi}\mathbf{e}_\varphi = \tan\theta\,\mathbf{e}_\theta\,,$$

and where the complex structure J (corresponding to a rotation in the tangent space) is defined by

$$J\mathbf{e}_\varphi = \mathbf{e}_\theta, \qquad J\mathbf{e}_\theta = -\mathbf{e}_\varphi\,.$$

At the level of the vorticity, the Euler equation becomes[‡]

$$D_t(\Omega + 2\omega \sin\theta) = 0\,, \tag{5}$$

and has to be complemented with the Biot-Savart law, which recovers at every instant t the stream function (and thus the velocity field) from the vorticity:

$$\psi(\xi_0) = \iint_{\mathbb{S}^2} \mathcal{G}(\xi,\xi_0)\Omega(\xi)\,\mathrm{d}\sigma(\xi)\,,$$

where $\mathcal{G}(\xi,\xi_0) = \frac{1}{2\pi}\ln\left(\frac{|\xi-\xi_0|}{2}\right)$, with $|\xi-\xi_0|$ the distance in $\mathbb{R}^3$ between $\xi \neq \xi_0$ on $\mathbb{S}^2$, is the Green function, satisfying

$$\Delta\mathcal{G}(\xi,\xi_0) = \delta(\xi-\xi_0) - \frac{1}{4\pi}\,, \tag{6}$$

with δ the Dirac delta distribution corresponding to a point vortex located at $\xi_0 \in \mathbb{S}^2$. Note that since the velocity field is divergence-free, an immediate consequence of the divergence theorem is the validity of the Gauss constraint

$$\iint_{\mathbb{S}^2} \Omega\,\mathrm{d}\sigma = 0\,, \tag{7}$$

so that the factor $-\frac{1}{4\pi}$ in (6) plays the role of a compensating uniform vorticity distribution on $\mathbb{S}^2$ to guarantee the validity of (7).

[‡]Sometimes termed the two-dimensional baroclinic Ertel equation for the material conservation of potential vorticity.

Generally, insight in the flow dynamics is more readily available working with the stream function ψ, rather than with the vorticity $\Omega = \Delta\psi$. Equation $(\mathcal{E}_\omega)$ is the barotropic vorticity equation, describing the motion of an inviscid, unforced, incompressible, homogeneous fluid on a rotating sphere.

- With respect to the symplectic structure on $\mathbb{S}^2$, whose Poisson bracket is given in spherical coordinates by

$$\{f, h\} = \frac{1}{\cos\theta}\left(\partial_\theta h\,\partial_\varphi f - \partial_\theta f\,\partial_\varphi h\right),$$

  the vorticity equation $(\mathcal{E}_\omega)$ can be expressed as the Hamiltonian flow

$$\partial_t(\Delta\psi + 2\omega\sin\theta) = \{\Delta\psi + 2\omega\sin\theta, \psi\}\,.$$

- Conservation laws and energy estimates for equation $(\mathcal{E}_\omega)$ are very similar to the more classical framework of the (two-dimensional) Euclidean space, or of the torus. In particular, one can use energy methods to prove local well-posedness in H^s, $s > 2$, and an analogue of the Beale-Kato-Majda theorem ensures global well-posedness in H^s, $s > 2$.

Symmetries The Euler equations $(\mathcal{E}_\omega)$ with different rotation speeds ω are related through the change-of-frame

$$\psi_0(\varphi,\theta,t) \longleftrightarrow \psi_\omega(\varphi,\theta,t) = \psi_0(\varphi+\omega t,\theta,t) + \omega \sin\theta$$

(i.e., ψ_ω solves $(\mathcal{E}_\omega) \Leftrightarrow \psi_0$ solves the Euler equation on a fixed sphere).

- (Scaling with $\lambda > 0$): $\psi(\varphi,\theta,t)$ solves $(\mathcal{E}_\omega)$, $\Leftrightarrow \lambda\psi(\varphi,\theta,\lambda t)$ solves $(\mathcal{E}_{\lambda\omega})$.
- Another invariance is related to the symmetries of the 2-sphere, given by the orthogonal group $\mathbb{O}(3)$, a compact Lie group of dimension 3, consisting of the isometries of $\mathbb{R}^3$ which fix the origin: one can think of $\mathbb{O}(3)$ as the group of orthogonal real 3×3 matrices or as a group of transformations of $\mathbb{R}^3$. The action of $\mathbb{O}(3)$ is defined by

$$Gf(X) = f(GX)\,, \qquad X\in\mathbb{S}^2\,, \quad G\in\mathbb{O}(3)\,,$$

for a scalar function $f:\mathbb{S}^2\to\mathbb{R}$ and the following transformations leave the set of solutions of $(\mathcal{E}_\omega)$ invariant§:

$$\begin{cases} \psi(X,t)\mapsto\psi(GX,t)\,, \\ U(X,t)\mapsto GU(GX,t)\,, \qquad X\in\mathbb{S}^2\,. \\ \Omega(X,t)\mapsto\Omega(GX,t)\,, \end{cases}$$

§ The non-abelian subgroup of $\mathbb{O}(3)$ of all orthogonal 3×3 real matrices R with $\det(R)=1$, itself a compact Lie group of dimension 3, is called the rotation group $\mathbb{SO}(3)$ since each transformation $X\mapsto RX$ with $R\in\mathbb{SO}(3)$ can be obtained by first choosing a fixed direction through the origin and subsequently rotating the coordinate system through a suitable angle about this direction as an axis.

Integrals of motion The following quantities are constant in time for smooth solutions of $(\mathcal{E}_\omega)$:

(i) (kinetic energy) $\frac{1}{2}\iint_{\mathbb{S}^2} |U|^2 \,\mathrm{d}\sigma$,

(ii) (Casimir invariants) $\iint_{\mathbb{S}^2} F(\Omega + 2\omega \sin\theta)\,\mathrm{d}\sigma$ for any differentiable function $F : \mathbb{R} \to \mathbb{R}$,

(iii) (first eigenspace of the Laplace-Beltrami operator) $\mathrm{e}^{\mathrm{i}m\omega t} c_1^m(t)$ for the coefficients

$$c_1^m = \iint_{\mathbb{S}^2} \Omega\, \overline{Y_1^j}\,\mathrm{d}\sigma\,, \qquad m \in \{-1, 0, 1\}\,,$$

of the $L^2(\mathbb{S}^2)$-expansion of the vorticity Ω in terms of the spherical harmonics $\{Y_j^m\}_{j\geq 1,\,|m|\leq j}$. In particular, the real number $c_1^0(t)$ and the absolute values of the complex numbers $c_1^{\pm 1}(t)$ are flow-invariants. For a non-rotating sphere $(\omega = 0)$, the vorticity components in the direction of each of the three spherical harmonics of degree 1 are invariant.

Stationary solutions Stationary solutions of $(\mathcal{E}_\omega)$ satisfy

$$[-\partial_\theta\psi\,\partial_\varphi + \partial_\varphi\psi\,\partial_\theta]\,(\Delta\psi + 2\omega\sin\theta) = 0\,. \tag{8}$$

Geometrically, (8) means that the gradients of the stream function ψ and of the potential vorticity $\Delta\psi + 2\omega\sin\theta$ are parallel. Since the gradient is orthogonal to the level set, in regions of $\mathbb{S}^2$ where $\operatorname{grad}\psi \neq (0,0)$ the rank theorem ensures that (8) is locally equivalent to the elliptic problem

$$\Delta\psi + 2\omega\sin\theta = F(\psi) \tag{9}$$

for some C^1-function F. It is easy to check that any solution of the elliptic problem (9) on $\mathbb{S}^2$ will also solve (8), but the converse is not true in general¶.

Two classes of explicit solutions of (8) are known:

- zonal solutions $\psi(\theta)$;
- for $F(s) = -j(j+1)s$, Rossby-Haurwitz waves of the form

$$\psi(\varphi,\theta) = \frac{2\omega}{2-j(j+1)}\sin\theta + \beta\,Y(\varphi,\theta)\,, \qquad j\geq 2\,, \quad \beta\in\mathbb{R}\,, \quad Y\in\mathbb{E}_j\,, \tag{10}$$

  where $\mathbb{E}_j$ is the $(2j+1)$-dimensional eigenspace of the Laplace-Beltrami operator associated to the eigenvalue $-j(j+1)$.

¶ For example, any zonal function ψ solves (8) but does not have to be a solution of (9), as shown by the case of constant functions.

Using the invariance properties, from the stationary Rossby-Haurwitz waves (10) one obtains explicit non-trivial travelling-wave solutions of $(\mathcal{E}_\omega)$ of the form

$$\psi(\varphi - ct, \theta) = \alpha \sin\theta + \beta\, Y(\varphi - ct, \theta) \tag{11}$$

with

$$Y \in E_j \ (j \geq 1)\,, \quad \alpha \in \mathbb{R}\,, \quad \beta \in \mathbb{R} \setminus \{0\}\,, \quad c = \frac{2\omega}{j(j+1)} + \alpha\,\frac{j(j+1)-2}{j(j+1)}\,,$$

solving

$$\Delta\psi + 2\omega \sin\theta = F(\psi)$$

for $F(\psi) = -j(j+1)$. Two particular cases are of great interest:

- for $\alpha = \frac{2\omega}{2-j(j+1)}$ with $j \geq 2$ we obtain the stationary waves (10), which propagate with wave speed $c = 0$;
- for $\alpha = \omega$ we obtain geostationary waves that propagate azimuthally with wave speed $c = \omega$, which can be subsumed into 3D stratospheric flows (as already pointed out).

Rigidity results

Theorem *Let ψ be a classical solution of*

$$\Delta\psi + 2\omega \sin\theta = F(\psi)$$

for some $\omega \in \mathbb{R}$. If $F' > 0$, then ψ is constant. If $F' > -6$, then ψ is a zonal flow, modulo a rotation in $\mathbb{O}(3)$.

- Note that the -6 is the second eigenvalue of the Laplace-Beltrami operator. Results of this type are available for general Riemannian surfaces, under the condition that F' is larger than the smallest eigenvalue of the Laplace-Beltrami operator ($F' > -2$ for the sphere). The symmetric structure of the sphere explains the improvement.
- The result is optimal since the Rossby-Haurwitz wave

$$\psi(\varphi,\theta) = -\tfrac{\omega}{2}\sin\theta + \beta\, Y(\varphi,\theta)\,, \qquad \beta \in \mathbb{R}\setminus\{0\}\,,$$

with Y a non-trivial linear combination of all spherical harmonics $\{Y_2^m\}_{|m|\le 2}$ satisfies $\Delta\psi + 2\omega\sin\theta = -6\psi$.

Using the invariance properties, this rigidity result permits us to find nonlinearities F for which the equation

$$\Delta\psi + 2\omega \sin\theta = F(\psi) \tag{13}$$

admits classical non-zonal solutions, starting from a non-trivial zonal solution and applying a rotation that transforms rotations about the polar axis into rotations about a fixed horizontal axis.

EXAMPLE 1 One can check that for every $\varepsilon > 0$ the function

$$\psi_0(\theta) = \ln\left(\frac{1+\varepsilon\sin\theta}{1-\varepsilon\sin\theta}\right), \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

is a zonal solution of (9) with $\omega = 0$, for

$$F(\psi) = -\frac{1-\varepsilon^2}{2}\left[2\sinh(\psi) + \sinh(2\psi)\right].$$

Consequently,

$$\psi(\varphi,\theta) = \ln[1 + \varepsilon\cos^2(\theta)\sin^2(\varphi - \varphi_0)]$$

is a non-zonal solution for every fixed $\varphi_0 \in [0, 2\pi)$. □

EXAMPLE 2 The zonal solution

$$\psi_0(\theta) = e^{\varepsilon \sin\theta} - 1\,, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\,,$$

for $F(\psi) = \varepsilon^2(1+\psi) - (1+\psi)\ln^2(1+\psi) - 2(1+\psi)\ln(1+\psi)$ and $\omega = 0$ leads to the non-zonal solution

$$\psi(\varphi, \theta) = e^{\varepsilon \cos\theta \sin(\varphi - \varphi_0)} - 1 \quad \text{with} \quad \varphi_0 \in [0, 2\pi) \quad \text{fixed.}$$

□

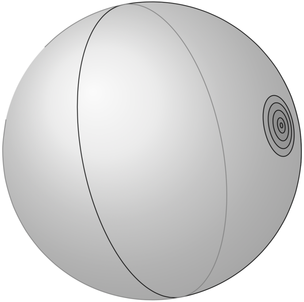

Figure: The streamlines of the solution in Example 2 are circles with collinear centres along a segment lying in the equatorial plane and passing through the centre of the sphere, since by passing to spherical coordinates in $\mathbb{R}^3$ we see that the level sets $[\cos\theta \sin(\varphi - \varphi_0) = d]$ are precisely the points on the sphere at distance $|d|$ from the line obtained rotating the x-axis by φ_0-degrees in the equatorial plane.

Stability of zonal flows

Consider a smooth zonal flow $\psi_0 = f(\theta)$, with associated azimuthal velocity and vorticity given by

$$U_0 = -f'(\theta)\mathbf{e}_\varphi\,, \qquad \Omega_0 = f''(\theta) - \tan\theta\, f'(\theta) = g(\theta)\,.$$

Theorem *If there exist parameters ϵ, $A \in \mathbb{R}$ such that*

$$\left|\frac{f'(\theta) - A\cos\theta}{g'(\theta)}\right| > \epsilon > 0 \qquad \textit{on} \qquad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

then the zonal flow is stable in $H^2(\mathbb{S}^2)$: if $\widehat{\psi}(t)$ is the solution of the vorticity equation with initial data $\psi(0) = \widehat{\psi}_0$, then $\|\widehat{\psi}(t) - \psi_0\|_{H^2} \lesssim \|\widehat{\psi}_0 - \psi_0\|_{H^2}$.

Proof. We implement Arnold's method by considering the functional

$$\mathcal{E}(\psi) = \iint_{\mathbb{S}^2} \left[\frac{1}{2}|U|^2 + K(\Omega) + A\sin\theta\,\Omega\right] \mathrm{d}\sigma\,,$$

which is a sum of conserved quantities for the flow. Choosing K such that $-f'(\theta) + K''(g(\theta))g'(\theta) = -A\cos\theta$, the second variation of $\mathcal{E}$ is then

$$d^2\mathcal{E}_{\psi_0}(\delta\psi) = \iint_{\mathbb{S}^2} \left[|\delta U|^2 + \frac{f'(\theta) - A\cos\theta}{g'(\theta)}(\delta\Omega)^2\right] \mathrm{d}\sigma\,. \qquad \square$$

Leu us discuss the zonal wind profiles of Uranus and Neptune, consisting of one broad retrograde equatorial jet flanked by two prograde jets at higher latitudes. The zonal flow is symmetric about the Equator for both planets, but there are noticeable differences of the latitudinal flow profiles:

- on Uranus the equatorial jet is located within the latitude band between 30°N and 30°S, while on Neptune it extends over 50°;
- the prograde/retrograde (eastward/westward) zonal flows on Uranus, measured relative to the planet's rotation speed about its polar axis, peak at about 200 m/s, respectively at 80 m/s, the corresponding values for Neptune being about 200 m/s and 400 m/s, respectively.

If the latitudinal profile of the zonal flow with respect to the rotation about the planet's polar axis (with zonal velocity $\theta \mapsto \omega \cos\theta$) is given by the function

$$U_0(\theta) = \alpha \cos^5\theta + \beta \cos^3\theta + \gamma \cos\theta\,, \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

for some real constants $\alpha \neq 0$, β and γ, we can now compute

$$\frac{-f'(\theta) + A\cos\theta}{g'(\theta)} = \frac{\cos^4\theta + \frac{\beta}{\alpha}\cos^2\theta + \frac{\gamma-\omega+A}{\alpha}}{30\left(\cos^4\theta + \frac{2(\beta-2\alpha)}{5\alpha}\cos^2\theta + \frac{\gamma-\omega-8\beta}{15\alpha}\right)}\,, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

so that the stability criterion applies if the quadratic polynomials (in $x = \cos^2\theta$)

$$x^2 + \frac{\beta}{\alpha}x + \frac{\gamma-\omega+A}{\alpha} \quad \text{and} \quad x^2 + \frac{2(\beta-2\alpha)}{5\alpha}x + \frac{\gamma-\omega-8\beta}{15\alpha}$$

have the same roots in the interval $(0,1)$.

- For Uranus the second quadratic polynomial is

$$x \mapsto x^2 - \tfrac{5}{2}\, x + \tfrac{155}{96}\,,$$

  with no real roots. Choosing $A \in \mathbb{R}$ so that the first quadratic polynomial has no roots, we conclude that the zonal flow pattern of Uranus is stable.
- For Neptune the second quadratic polynomial is

$$x \mapsto x^2 - \tfrac{211}{160}\, x + \tfrac{20731}{30720}\,,$$

  with no real roots. Consequently, choosing $A \in \mathbb{R}$ so that the first quadratic polynomial has no roots, we conclude that the zonal flow pattern of Neptune is also stable.

Remark: The zonal jet patterns on Jupiter and Saturn are not far off from entering the framework of this stability result. However, the profiles of terrestrial stratospheric jets are well beyond the condition that we require, as is to be expected since the Earth's polar jet stream is known to be unstable.

Consider a stationary solution ψ_0. We implement Arnold's method by defining

$$\mathcal{E}(\psi) = \iint_{\mathbb{S}^2} \left[\frac{1}{2}|U_0 + U|^2 + K(\Omega_0 + \Omega + 2\omega \sin\theta) + A \sin\theta\Omega + B(\mathbb{P}_1\psi)^2\right] \mathrm{d}\sigma$$

as a sum of conserved quantities for the flow ($\mathbb{P}_k$ being the projection on $\mathbb{E}_k$). Choosing

$$K''(F(x))F'(x) = 1 \quad \text{and} \quad A = \omega\,,$$

we ensure that the first variation is zero. The second variation $d^2\mathcal{E}_0(\delta\psi)$ is

$$\iint_{\mathbb{S}^2} \left[\left(2 + \frac{4}{F'(\psi_0)} + 2B\right)(\mathbb{P}_1\delta\psi)^2 + \sum_{k\geq 2}\left(k(k+1) + \frac{k^2(k+1)^2}{F'(\psi_0)}\right)(\mathbb{P}_k\delta\psi)^2\right] \mathrm{d}\sigma\,.$$

The question of the coercivity of $d^2\mathcal{E}_0$ reduces to determining eigenvalues of the Schrödinger operator $-\frac{\Delta}{F'(\psi)} + 1$ on the orthogonal complement of $\mathbb{E}_1$. We distinguish several cases, according to the the range of $F'(\psi_0)$:

- If $F' > 0$, the quadratic form is positive-definite if one chooses $B = 0$. By the rigidity result, $F' > 0$ forces constant solutions.
- If $-6 < F'(\psi_0) < 0$, then the quadratic form is negative-definite: indeed, $k(k+1) + \frac{k^2(k+1)^2}{F'(\psi_0)} > 0$ for all $k \geq 2$, and the mode $k = 1$ can be handled by choosing $B = -10$. By the rigidity result, the solutions are zonal up to a rotation.

The above considerations yield the following result.

Theorem *For* $0 > F' > -6$*, the stationary solutions are stable in* $H^2(\mathbb{S}^2)$*.*

This result applies to the explicit stationary solutions discussed in Example 1 and in Example 2. Indeed, since

$$\frac{1}{F'(\psi)} = -\,\frac{1}{2(1-\varepsilon^2)}\,\frac{[1-\varepsilon^2\cos^2(\theta)\sin^2(\varphi-\varphi_0)]^2}{1+3\,\varepsilon^2\cos^2(\theta)\sin^2(\varphi-\varphi_0)}\,,$$

and

$$\frac{1}{F'(\psi_0)} = -\,\frac{1}{2-\varepsilon^2+4\varepsilon\cos(\theta)\sin(\varphi-\varphi_0)+\varepsilon^2\cos^2(\theta)\sin^2(\varphi-\varphi_0)}\,,$$

respectively, we see that for $\varepsilon > 0$ small enough these solutions are stable.

Note that with the exception of the equatorial regions containing a broad eastward zonal jet, vortices are generally found on Jupiter and Saturn at all latitudes, preferentially in regions of westward zonal flow.

The limiting case in the above stability theorem is given by $F' = -6$, which corresponds to Rossby-Haurwitz solutions in $\mathbb{E}_1 + \mathbb{E}_2$. These solutions, with a more intricate latitude variation, are of considerable interest in meteorology. For example, the wave obtained by setting Y proportional to the spherical harmonics Y_2^1 is commonly observed in the terrestrial atmosphere, being known as the 5-day wave since it travels westwards with a period of about 5 days. These waves are also preponderant in the atmospheres of the outer planets of our solar system (Jupiter, Saturn, Uranus, Neptune). The instability of the Rossby-Haurwitz waves is a key factor in the lack of predictability of the weather in long-term forecasts and earlier attempts to study the stability of the degree 2 waves by numerical means are somewhat inconclusive, the results being partly contradictory[‖].

[‖] See the discussion in P. Bénard, Stability of Rossby-Haurwitz waves, *Quart. J. Roy. Met. Soc.* **146** (2020), 613–628.

Theorem *(i) Non-zonal Rossby-Haurwitz waves of the form*

$$\psi_0(t) = \alpha \sin\theta + \beta\, Y(\varphi - ct, \theta)$$

with $Y \in E_j$ $(j \geq 1)$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R} \setminus \{0\}$, $c = \frac{2\omega}{j(j+1)} + \alpha\, \frac{j(j+1)-2}{j(j+1)}$, *are unstable: there exists* $\epsilon > 0$ *and a sequence* $\widehat{\psi}_0^n \to \psi_0$ *in* $H^2(\mathbb{S}^2)$, *so that for the solutions* $\widehat{\psi}^n(t)$ *of* $(\mathcal{E}_\omega)$ *with initial data* $\widehat{\psi}^n(0) = \widehat{\psi}_0^n$ *we have*

$$\sup_{t>0} \|\psi^n(t) - \psi_0(t)\|_{L^2(\mathbb{S}^2)} > \epsilon > 0\,.$$

(ii) The zonal Rossby-Haurwitz flows

$$\psi_0(\theta) = \alpha \sin\theta + \beta\, Y_2^0(\theta)\,, \qquad \alpha \in \mathbb{R}\,, \quad \beta \in \mathbb{R} \setminus \{0\}\,,$$

of degree $n \leq 2$ *are stable in* $H^2(\mathbb{S}^2)$*.*

Proof. For (i), we estimate $\sup_{t>0} \|\widehat{\psi}^n(t) - \psi(t)\|^2_{L^2(\mathbb{S}^2, \mathrm{d}\sigma)}$ from below, for

$$\widehat{\psi}^n(t) = \left(\alpha + \tfrac{1}{n}\right) \sin\theta + \beta\, Y(\varphi - \widehat{c}\, t, \theta)\,, \quad \psi_0(t) = \alpha \sin\theta + \beta\, Y(\varphi - ct, \theta)\,,$$

with $c = \frac{2\omega}{j(j+1)} + \alpha\, \frac{j(j+1)-2}{j(j+1)}$ and $\widehat{c} = \frac{2\omega}{j(j+1)} + \left(\alpha + \frac{1}{n}\right)\, \frac{j(j+1)-2}{j(j+1)}$.

To prove (ii), note that, using the scaling and change-of-frame symmetries, it suffices to consider the case $\omega = 0$, $\beta = 1$. Consider a smooth perturbation

$$\psi(\varphi, \theta, t) = \sum_{l=1}^{\infty} \Big\{ \sum_{m=-l}^{l} c_l^m(t)\, Y_l^m(\varphi, \theta) \Big\}$$

of the zonal flow, expressed in terms of the spherical harmonics Y_l^m by means of the time-dependent coefficients $c_l^m(t) \in \mathbb{C}$. Since ψ is real-valued, we have

$$c_l^{-m}(t) = \int_{\mathbb{S}^2} \psi\, \overline{Y_l^{-m}}\, d\sigma = (-1)^m \int_{\mathbb{S}^2} \psi\, Y_l^m\, d\sigma = (-1)^m \overline{\int_{\mathbb{S}^2} \psi\, \overline{Y_l^m}\, d\sigma} = (-1)^m\, \overline{c_l^m(t)}\,,$$

Furthermore, we know that $c_1^0(t) = c_1^0(0)$ and $|c_1^{\pm 1}(t) = |c_1^{\pm 1}(0)|$ for $t \geq 0$. The conservation of energy and the time-invariance of $\int_{\mathbb{S}^2} |\Delta\psi|^2\, \mathrm{d}\sigma$ give

$$\sum_{l=3}^{\infty} [l^2(l+1)^2 - 6l(l+1)] \Big\{ \sum_{m=-l}^{l} |c_l^m(t)|^2 \Big\} < \varepsilon^2\,, \qquad t \geq 0\,.$$

if $\|\Delta[\psi_0 - \psi(\cdot, 0)]\|_{L^2} < \varepsilon$. Thus the instability can only be caused by a substantial energy transfer between the spherical harmonic components of mode $l = 2$. To rule this out, we rely on the time-invariance of the integrals

$$I_k(\psi(\cdot, t)) = \int_{\mathbb{S}^2} \Big(\Delta\psi(\cdot, t)\Big)^k\, \mathrm{d}\sigma\,, \qquad k \in \{2, 3, 5\}\,.$$