Nonlinear forward-backward problems

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Abel Symposium
June 13th - 16h, 2023

European Research Council
Established by the European Commission

institut universitaire de France

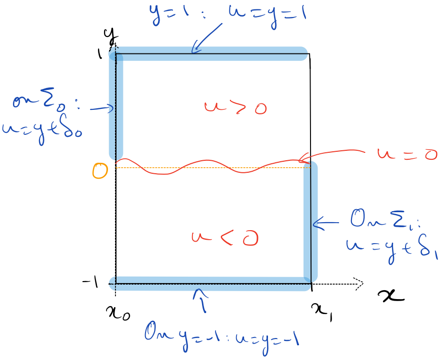
Outline

Presentation of the problem

Goal: construct solutions of

$$(\text{FB}) \qquad \begin{aligned} &uu_x - u_{yy} = f \quad \text{in } (x_0, x_1) \times (-1, 1) =: \Omega, \\ &u_{|y=\pm 1} = \pm 1, \quad u_{|\Sigma_i} = y + \delta_i, \end{aligned}$$

for $\|f\|, \|\delta_i\| \ll 1$, i.e. $\|u - y\| \ll 1$: Changing-sign solution.



Zone $\{u > 0\}$: forward parabolic equation.
Zone $\{u < 0\}$: backward parabolic equation.
→ Boundary data on blue part of boundary.

⚠ Line $\{u = 0\}$ is a free boundary!

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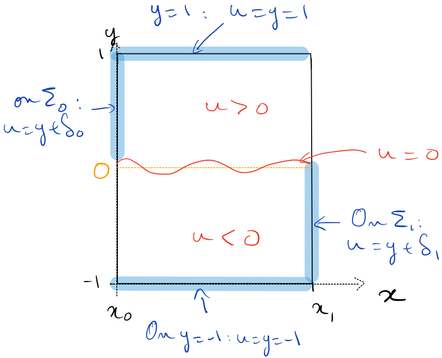

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Motivation: boundary layer separation

Original motivation: recirculation in Prandtl system

$$\begin{aligned} uu_x + vu_y - u_{yy} &= f(x), \\ u_x + v_y &= 0. \end{aligned}$$

- Case $u > 0$: equation is forward parabolic (Oleinik, 1962);
- Changing sign solutions occur after boundary layer separation.

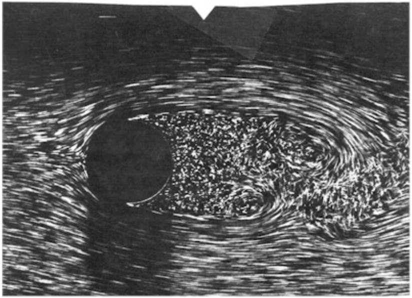

Figure: Cross-section of a flow past a cylinder (source: ONERA, France)

Reversed flow within Prandtl system

Difficulties:

- **Nonlocality** of the Prandtl equation: even the linearized equation is difficult to solve.
- Potential **loss of derivatives** and need for high regularity estimates to construct a strong solution.

First result of [Iyer & Masmoudi, '22]: *a priori* estimates for the Prandtl system with reversal flow.
However, this is not sufficient to construct a solution!
$\rightarrow$ Second result by same authors using results & methodology of present talk to obtain existence of solutions.

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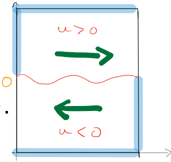
A first naïve attempt



$$uu_x - u_{yy} = f \quad \text{in } \Omega, \quad u_{|y=\pm 1} = \pm 1, \quad u_{|\Sigma_i} = y + \delta_i.$$

First observation: importance of the geometry of $\{u = 0\}$.

Idea: define $u = y + \lim_{n\to\infty} u_n$, where

$$(y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f,$$
$$u_{n+1|\Sigma_i} = \delta_i, \quad u_{n+1|y=\pm 1} = 0.$$

Strong a priori estimate:

$$(1) \qquad \|u_n\|_X \lesssim \|f\| + \|\delta_0\| + \|\delta_1\| =: \eta \ll 1,$$

where X is a "strong" space, controlling $W^{1,\infty}$.

Weak geometric bound: in a weaker space Y, say $Y = L^2_x H^1_y$,

$$\|u_{n+1} - u_n\|_Y \lesssim \eta \|u_n - u_{n-1}\|_Y.$$

Conclusion: $(u_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in Y for $\eta \ll 1$.

But (1) does not hold in general!

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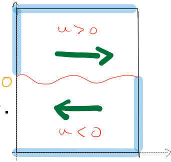

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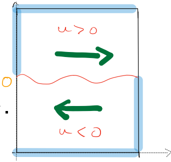

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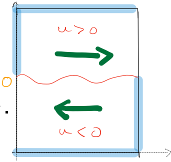

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But (1) does not hold in general!

Why does the previous attempt fail?

Consider the linear equation

$$
(2) \qquad \begin{aligned} y\partial_x v - \partial_{yy} v = f \quad \text{in } \Omega, \\ v_{|y=\pm 1} = 0, \quad v_{|\Sigma_i} = \delta_i \end{aligned}
$$

with $f \in C_c^\infty(\Omega)$, $\delta_i \in C_c^\infty(\Sigma_i)$.

Fact #1: (2) has a unique weak solution $v \in H_x^{2/3} L_y^2 \cap L_x^2 H_y^2$.

Fact #2: This solution is not smooth in general!

Fact #3: $v \in H_x^{5/3} L_y^2 \cap L_x^2 H_y^5$ if and only if

(f, δ_0, δ_1) satisfy two orthogonality conditions.

Remark: similar to elliptic equations in domains with corners...

Main result

Theorem: [D., Marbach, Rax, 2022]
Assume that $\|\delta_i\|, \|f\|$ are small.
There exists a manifold $\mathcal{M}$ of codimension 2 s.t.

$$u\partial_x u - \partial_{yy} u = f \quad \text{in } (x_0, x_1) \times (-1, 1),$$
$$u_{|\Sigma_i} = y + \delta_i, \quad u_{|y=\pm 1} = \pm 1,$$

has a solution in $H_x^{5/3} L_y^2 \cap L_x^2 H_y^5$ iff $(f, \delta_0, \delta_1) \in \mathcal{M}$.

Remarks:

- Nonlinear orthogonality conditions. Depend on solution itself;
- Difficulty is NOT the derivation of *a priori* estimates;
- Extended to Prandtl by [Iyer&Masmoudi, 2022];
- Control of k derivatives in $x \Rightarrow 2k$ orthogonality conditions $\Rightarrow$ Manifold of codimension $2k$.

Scheme of proof

1. **Linearized equation:** for $\bar{u}$ smooth s.t. $\|\bar{u} - y\| \ll 1$,

$$\bar{u}\partial_x u - \partial_{yy} u = f, \quad u_{|\pm 1} = 0, \; u_{|\Sigma_i} = \delta_i.$$

   - Existence/uniqueness of weak solutions;
   - Orthogonality conditions for strong solutions (depend on $\bar{u}$!)

2. **Modified iterative scheme:**

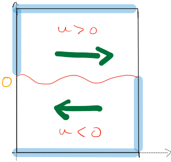
$$(y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1},$$
$$u_{n+1|\Sigma_i} = \delta_i^{n+1}, \quad u_{n+1|y=\pm 1} = 0.$$

   where $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ ensure that orthogonality conditions are satisfied.

3. **Uniform estimate** on u_n in strong space X (easy).
4. Dependency of orthogonality conditions on $\bar{u}$ in order to have a **Cauchy sequence** in weak space Y (difficult).

Outline

Review of results on the linear shear flow model



(LSF)

$$y\partial_x v - \partial_{yy} v = f \quad \text{in } \Omega,$$
$$v_{|y=\pm 1} = 0, \quad v_{|\Sigma_i} = \delta_i.$$

Weak solutions:

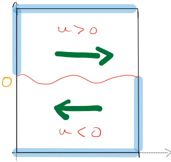
- If $f \in L^2_x(H^{-1}_y)$, $|y|^{1/2}\delta_i \in L^2$: existence [Fichera,60] and uniqueness [Baouendi-Grisvard,68] of solutions $v \in L^2_x H^1_z$.
- If $f \in L^2$, $|y|^{1/2}\delta_i \in H^1_0(\Sigma_i)$: $v \in H^{2/3}_x L^2_y \cap L^2_x H^2_y$ [Pagani,75].

Observation: (LSF) is stable by differentiation w.r.t. x !

Natural guess: if $f \in H^1_x L^2_y$ ($+$ conditions on δ_i), then $v \in H^{5/3}_x L^2_y \cap H^1_x H^2_y$.

$\rightarrow$ FALSE without additional orthogonality conditions.

Review of results on the linear shear flow model



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Observation: (LSF) is stable by differentiation w.r.t. x !
Natural guess: if $f \in H^1_x L^2_y$ (+ conditions on δ_i), then $v \in H^{5/3}_x L^2_y \cap H^1_x H^2_y$.

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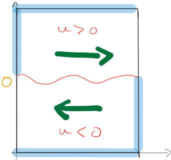
Orthogonality conditions for higher regularity solutions

If $f \in H^1_x L^2_y$ with $f_{|x=x_i} = \delta_i = 0$, then v_x solves



$$(3) \qquad \begin{aligned} y\partial_x v_x - \partial_{yy} v_x = f_x \quad \text{in } \Omega, \\ v_{x|y=\pm 1} = 0, \quad v_{x|\Sigma_i} = 0. \end{aligned}$$

(Tentative) reconstruction: $\exists! w \in H^{2/3}_x L^2_y \cap L^2_x H^2_y$ sol. of (3);

Let

$$\tilde{v}(x,y) = \begin{cases} \int_{x_0}^{x} w(x',y)\,dx' & \text{if } y > 0, \\ -\int_{x}^{x_1} w(x',y)\,dx' & \text{if } y < 0. \end{cases}$$

Then...

- $\partial_x \tilde{v} = w$;
- $\tilde{v}$ solves (LSF) in $\Omega \setminus \{y = 0\}$.

BUT in general $\tilde{v}$ and $\partial_y \tilde{v}$ have a jump across $y = 0$.

$$[\tilde{v}]_{|y=0} = [\partial_y \tilde{v}]_{|y=0} = 0 \iff \underbrace{\int_{x_0}^{x_1} w(x,0)dx = \int_{x_0}^{x_1} \partial_y w(x,0)dx}_{\text{linear forms of } (f, \delta_0, \delta_1)} = 0.$$

Decomposition of the solution into singular profiles

Polar-like variables near $(x_i, 0)$: let

$$r_i = \left(|x - x_i|^{2/3} + y^2\right)^{1/2}, \quad t_i = \frac{y}{|x - x_i|^{1/3}}$$

Lemma: *for any $k \in \mathbf{Z}$, there exists $G_k \in C_b^\infty(\mathbf{R})$ such that*

$$U_{k,i} = r_i^{\frac{1}{2}+3k} G_k(t_i)$$

is a solution of $y\partial_x U - \partial_{yy} U = 0$ in $\mathbf{R}^2_\pm$.

Consequence: if (f, δ_0, δ_1) are smooth, there exist $(c_0, c_1) \in \mathbf{R}^2$ such that

$$v = c_0 U_{0,0} + c_1 U_{0,1} + v_{\text{reg}}, \quad \text{with } v_{\text{reg}} \in H_x^{5/3} L_y^2 \cap H_x^1 H_y^2.$$

Remarks:

- Orthogonality conditions $\iff$ cancellation of c_0, c_1;
- Decomposition up to any order of regularity is possible.

Extension to non-constant coefficients

Consider now, for $\bar{u} \in H_x^{5/3} L_y^2 \cap L_x^2 H_y^5$,

$$\bar{u}\partial_x v - \partial_{yy} v = f \quad \text{in } \Omega, \qquad v_{|\pm 1} = 0, \quad v_{|\Sigma_i} = \delta_i. \tag{4}$$

All previous results can be extended when $\|\bar{u} - y\| \ll 1$:

- Existence/uniqueness of weak solutions when $f \in L^2$, $\delta_i \in H_0^1(\Sigma_i)$.
- Orthogonality conditions for higher regularity (2 conditions for each x derivative);
- Decomposition into sum of singular profiles + smooth remainder.

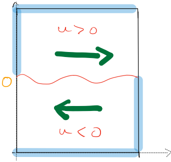
Remark: the orthogonality conditions depend on $\bar{u}$! Write as

$$\ell^1_{\bar{u}}(f, \delta_0, \delta_1) = \ell^2_{\bar{u}}(f, \delta_0, \delta_1) = 0$$

with $\ell^1_{\bar{u}}, \ell^2_{\bar{u}}$ two independent linear forms.

Outline

Iterative scheme (easy step)



Remember that

$$(y + u_n)\partial_x u_{n+1} - \partial_{yy} u_{n+1} = f^{n+1},$$
$$u_{n+1|\Sigma_i} = \delta_i^{n+1}, \quad u_{n+1|y=\pm 1} = 0.$$

Goal: choose $(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1})$ s.t.

$$\ell^j_{y+u_n}(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1}) = 0, \quad j = 1, 2.$$

Idea: take

$$(f^{n+1}, \delta_0^{n+1}, \delta_1^{n+1}) = (f, \delta_0, \delta_1) + \nu_1^{n+1} T_1 + \nu_2^{n+1} T_2,$$

where T_1, T_2 are fixed triplets such that

$$\ell^j_y(T_i) = \delta_{i,j}.$$

$\rightarrow$ Invert 2×2 matrix (close to identity).

Conclusion: uniform strong bound on u_n (in $X = H^{5/3}_x L^2_y \cap L^2_x H^5_y$).

Geometric bound in a "weak" space (difficult step)

The difference $w_n := u_{n+1} - u_n$ solves

$$(y+u_n)\partial_x w_n - \partial_{yy} w_n = -w_{n-1}\partial_x u_n + (\nu_1^{n+1} - \nu_1^n) f_1 + (\nu_2^{n+1} - \nu_2^n) f_2.$$

Question: $\|w_n\|_Y \lesssim \eta \|w_{n-1}\|_Y$?

→ Need to **find a space** Y such that:

- Nice product law: $\|w_n\|_Y \lesssim \|u_n\|_X \|w_{n-1}\|_Y + |\nu^{n+1} - \nu^n|$
- Lipschitz continuity of $\ell^j_{\bar{u}}$ in Y:
  $|\nu^{n+1} - \nu^n| \lesssim \eta \left\| \ell^j_{y+u_n} - \ell^j_{y+u_{n-1}} \right\| \lesssim \eta \|w_{n-1}\|_Y.$

Difficult!

- Analyze properties of $\bar{u} \mapsto \ell^j_{\bar{u}}$;
- Y = fractional space. Need to interpolate between closed subspaces of Sobolev spaces...

Remark: both difficulties are addressed by having an "explicit" representation of $\ell^j_{\bar{u}}$.

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The manifold $\mathcal{M}$

As $n \to \infty$, for any (f, δ_0, δ_1), one obtains a solution of

$$u\partial_x u - \partial_{yy} u = f + \nu_1 f_1 + \nu_2 f_2$$
$$u_{|y=\pm 1} = 0, \quad u_{|\Sigma_i} = \delta_i + \nu_1 \delta_{i,1} + \nu_2 \delta_{i,2},$$

for $\nu_1, \nu_2 \in \mathbf{R}$ ensuring orthogonality conditions.

$\rightarrow$ Not a solution of the original problem!

... Unless $\nu_1 = \nu_2 = 0$: cancellation of two scalar quantities, close to linear forms.

$\rightarrow$ Manifold of co-dimension two.

Summary

- Existence and uniqueness for quasilinear forward-backward parabolic equation, under orthogonality conditions;
- Rare case where *a priori* estimates don't provide a solution!
- Difficulty linked to quasilinear nature of the problem: orthogonality conditions depend on solution itself.
- Methodology could be adapted to other settings (e.g. nonlinear elliptic equations in domains with corners, traveling fronts in reaction-diffusion equations...)

Thank you for your attention!
Vi takker for oppmerksomheten!