Explicit formula and zero dispersion limit for the Benjamin–Ono equation

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The Benjamin–Ono equation

Long internal gravity waves in a two–layer fluid with infinite depth (Benjamin (1967), Davis–Acrivos (1967), Ono (1975))

$$\partial_t u = \partial_x(|D_x|u - u^2) \quad , \quad u = u(t,x) \in \mathbb{R} \, ,$$

$$u(t,x) \underset{x\to\infty}{\longrightarrow} 0 \, ,$$

$$\widehat{|D_x|f}(\xi) := |\xi|\hat{f}(\xi) \, , \; \xi \in \mathbb{R} \, .$$

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Global wellposedness : Saut (1979) in $H^2(\mathbb{R})$, ..., Tao (2004) in $H^1(\mathbb{R})$,..., Ionescu–Kenig (2007) in $L^2(\mathbb{R})$, see also Ifrim–Tataru (2017), Killip–Laurens–Vișan (2023) in $H^s(\mathbb{R}), s > -1/2$ using the Lax pair structure (Nakamura (1979), Fokas–Ablowitz (1983), Wu (2016)). With periodic boundary conditions : Molinet (2008) in $L^2(\mathbb{T})$, Kappeler–Topalov–PG (2020) in $H^s(\mathbb{T}), s > -1/2$, with counterexample on $H^{-1/2}(\mathbb{T})$.

The zero dispersion limit

Consider the Benjamin–Ono equation with small dispersion $\varepsilon > 0$,

$$\partial_t u^\varepsilon + \partial_x[(u^\varepsilon)^2] = \varepsilon \partial_x |D_x| u^\varepsilon \ , \ u^\varepsilon(0) = u_0 \ ,$$

Question : what is the limit of $u^\varepsilon(t)$ as $\varepsilon \to 0$?

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If $|t|$ is small enough, there exists a smooth solution to the inviscid Burgers–Hopf equation,

$$\partial_t u + \partial_x[u^2] = 0 \ , \ u(0,x) = u_0(x) \ ,$$

and $u^\varepsilon(t,x) \to u(t,x)$.

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Main question : What is the limit of $u^\varepsilon(t)$ after the time of shock formation ?

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We are going to answer this question in some wide generality.

Some references for the zero dispersion limit

Long standing problem, starting in the analogous question for the Korteweg–de Vries equation with Lax–Levermore (1983), Venakides (1985), Deift–Venakides–Zhou (1997), Clayes–Grava (2009),... studied the problem by using inverse scattering theory. Necessitates quite stringent assumptions on the datum u_0.

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Numerical approaches by Grava–Klein (2007,...) Fiksdal (Master thesis, NTNU, 2013).

Results for Benjamin–Ono by Miller–Xu (2011), Miller–Wetzel (2016), and more recently by Gassot (2021, 2022) in the periodic case, using the inverse spectral theory for bell–shaped data (typically $u_0(x) = 2(1 + x^2)^{-1}$ on the line).

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In this talk : we revisit the problem for the Benjamin–Ono equation, using a different approach, bypassing inverse scattering theory.

The method of characteristics

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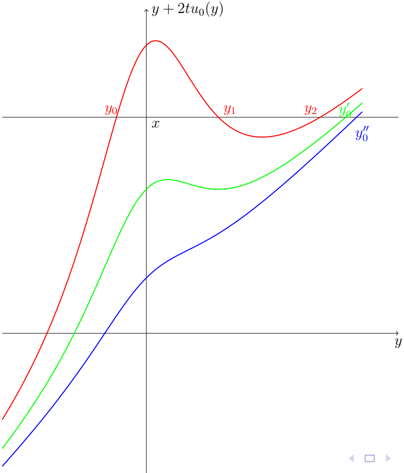

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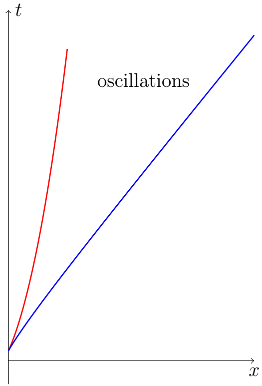

Main result

Theorem (PG, 2023)

For every $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *for every* $t \in \mathbb{R}$, *the solution* $u^\varepsilon(t)$ *of*

$$\partial_t u^\varepsilon + \partial_x[(u^\varepsilon)^2] = \varepsilon \partial_x |D_x| u^\varepsilon \ , \ \ u^\varepsilon(0,x) = u_0(x) \ ,$$

is weakly convergent in $L^2(\mathbb{R})$ *as* $\varepsilon \to 0$.
Furthermore, if u_0 *is a rational function, and if* (t,x) *is such that the algebraic equation* $y + 2tu_0(y) = x$ *has exactly* $2\ell + 1$ *real simple solutions* $y_0(t,x) < \cdots < y_{2\ell}(t,x)$, *then the weak limit is given by*

$$u(t,x) = \sum_{k=0}^{2\ell} (-1)^k u_0(y_k(t,x)) \ . \tag{1}$$

Remark : Formula (1) above is due to Miller–Wetzel in the special case of a bell–shaped rational potential.

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1. Prove an explicit formula for the solution of the Benjamin–Ono equation with arbitrary datum $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, bypassing inverse spectral theory (but using the Lax pair structure).

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1. Prove an explicit formula for the solution of the Benjamin–Ono equation with arbitrary datum $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, bypassing inverse spectral theory (but using the Lax pair structure).
2. Pass to the zero dispersion limit in the above formula, obtaining an explicit formula for the limit $u(t, x)$ for any $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

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1. Prove an explicit formula for the solution of the Benjamin–Ono equation with arbitrary datum $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, bypassing inverse spectral theory (but using the Lax pair structure).
2. Pass to the zero dispersion limit in the above formula, obtaining an explicit formula for the limit $u(t,x)$ for any $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
3. Link with the multivalued solution of the inviscid Burgers–Hopf equation from the method of characteristics.

The Lax pair

The Hardy space is

$$\begin{aligned} L^2_+(\mathbb{R}) &:= \{f \in L^2(\mathbb{R}) : \forall \xi < 0, \hat{f}(\xi) = 0\} \\ &= \{f \text{ holomorphic on } \mathbb{C}_+ : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < +\infty\} \end{aligned}$$

The associated Riesz–Szegő projector is

$$\Pi : L^2(\mathbb{R}) \to L^2_+(\mathbb{R}) \ , \ \widehat{\Pi f}(\xi) = \mathbf{1}_{\xi \geq 0} \hat{f}(\xi) \ .$$

Given $b \in L^\infty$, define the Toeplitz operator of symbol b,

$$T_b : L^2_+ \to L^2_+ \ , \ f \mapsto T_b f := \Pi(bf) \ .$$

Notice that $T_b^* = T_{\overline{b}}$.

The Lax pair, continued

For $u \in L^\infty$, real valued, define $L_u : H^1_+ := H^1 \cap L^2_+ \to L^2_+$ by

$$L_u(f) = \frac{1}{i}\frac{df}{dx} - T_u f \ .$$

L_u is unbounded selfadjoint on L^2_+ with $\mathrm{Dom}(L_u) := H^1_+ = H^1 \cap L^2_+$.
Also define, for $u \in H^2$, real valued

$$B_u := i(T_{|D|u} - T_u^2) \ .$$

Notice that $B_u : L^2_+ \to L^2_+$, $B_u : H^1_+ \to H^1_+$ and $B_u^* = -B_u$.

The Lax pair, statement

Theorem (Nakamura (1979), Fokas–Ablowitz (1983),Wu (2016), PG–Kappeler(2019))

If $u \in C(\mathbb{R}, H^2(\mathbb{R}))$ *solves the Benjamin–Ono equation, then*

$$\frac{dL_{u(t)}}{dt} = [B_{u(t)}, L_{u(t)}] \ .$$

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Corollary

Define the family of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ *by*

$$U'(t) = B_{u(t)}U(t) \ , \ U(0) = \mathrm{Id} \ .$$

Then

$$L_{u(t)} = U(t)L_{u(0)}U(t)^* \ .$$

The explicit formula

Consider the Lax–Beurling semigroup on $L^2_+(\mathbb{R})$, $S(\eta) := T_{e^{i\eta x}}$, $\eta \geq 0$.
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$$\begin{aligned} S(\eta)^* &= e^{-i\eta G}\ ,\ \eta \geq 0\ ,\ \widehat{Gf}(\xi) = i\frac{d\hat{f}}{d\xi}\mathbf{1}_{\xi>0}\ , \\ \mathrm{Dom}(G) &= \{f \in L^2_+(\mathbb{R}) : \hat{f}_{|]0,+\infty[} \in H^1(]0,+\infty[)\}\ . \end{aligned}$$

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Define $I_+(f) := \hat{f}(0^+)$ if $\hat{f}_{|]0,1[} \in H^1(]0,1[)$.

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Theorem (PG, 2022)

The solution $u \in C(\mathbb{R}, H^2(\mathbb{R}))$ *of the Benjamin–Ono equation with* $u(0) = u_0$ *is given by* $u(t,x) = \Pi u(t,x) + \overline{\Pi u(t,x)}, x \in \mathbb{R}$ *with*

$$\forall x \in \mathbb{C}_+ \ , \ \Pi u(t,x) = \frac{1}{2i\pi} I_+[(G - 2tL_{u_0} - x)^{-1}\Pi u_0] \ .$$

Sketch of the proof

First reformulate the inverse Fourier formula for $f \in L^2_+(\mathbb{R})$.

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With $\chi_\varepsilon(y) := (1 - i\varepsilon y)^{-1}$, we have, for every $x \in \mathbb{C}_+$,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\hat{f}(\xi)\,d\xi = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} f(x)\mathrm{e}^{-ix\xi}\overline{\chi_\varepsilon(x)}\,dx\,d\xi \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\lim_{\varepsilon\to 0^+}\langle f|S(\xi)\chi_\varepsilon\rangle\,d\xi \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\lim_{\varepsilon\to 0^+}\langle S(\xi)^* f|\chi_\varepsilon\rangle\,d\xi \\ &= \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{ix\xi}\lim_{\varepsilon\to 0^+}\langle \mathrm{e}^{-i\xi G} f|\chi_\varepsilon\rangle\,d\xi \\ &= \frac{1}{2i\pi}\lim_{\varepsilon\to 0^+}\langle (G-x)^{-1}f|\chi_\varepsilon\rangle = \frac{1}{2i\pi}I_+((G-x)^{-1}f)\ . \end{aligned}$$

Then apply this formula to $f = \Pi u(t)$ and deform it through $U(t)^*$.

A typical calculation

Since $U'(t) = B_{u(t)}U(t)$, we have

$$\frac{d}{dt}U(t)^*GU(t) = U(t)^*[G, B_{u(t)}]U(t)$$

Using

$$[G, B_u] = -2L_u + i[L_u^2, G] \ ,$$

and

$$U(t)^*L_{u(t)}U(t) = L_{u_0} \ ,$$

we conclude

$$U(t)^*GU(t) = -2tL_{u_0} + \mathrm{e}^{itL_{u_0}^2}G\mathrm{e}^{-itL_{u_0}^2} \ .$$

The zero dispersion limit

After rescaling $t \mapsto \varepsilon t$ and $u \mapsto u/\varepsilon$, we get, using $L_u = D - T_u$,

$$\begin{aligned}\forall x \in \mathbb{C}_+ \ , \ \Pi u^\varepsilon(t,x) &= \frac{1}{2i\pi} I_+[(G - 2\varepsilon t L_{u_0/\varepsilon} - x)^{-1}\Pi u_0] \\ &= \frac{1}{2i\pi} I_+[(G - 2\varepsilon t D + 2t T_{u_0} - x)^{-1}\Pi u_0] \ .\end{aligned}$$

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Using the L^2 conservation law and the strong convergence of the resolvents, we obtain $u^\varepsilon(t) \rightharpoonup u(t)$ in $L^2(\mathbb{R})$ with

$$\forall x \in \mathbb{C}_+ \ , \ \Pi u(t,x) = \frac{1}{2i\pi} I_+[(G + 2t T_{u_0} - x\mathrm{Id})^{-1}\Pi u_0]$$

The case of rational data

For simplicity, let us assume

$$u_0(y) = \frac{ay+b}{1+y^2}\ ,\ (a,b)\in\mathbb{R}^2\ ,\ \Pi u_0(y) = \frac{c}{y+i}\ .$$

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1. One real solution y_0 and two complex solutions $y_1, y_2 = \overline{y}_1$ with $\mathrm{Im}(y_2) > 0$.
2. Three real solutions $y_0 < y_1 < y_2$.

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Note that, if x is slightly shifted into the upper half plane with $\mathrm{Im}(x) > 0$ small, then

$$\mathrm{Im}(y_0) > 0\ ,\ \mathrm{Im}(y_1) < 0\ ,\ \mathrm{Im}(y_2) > 0\ ,$$

so the zeroes of $y \mapsto y - x + 2tu_0(y)$ in the upper half plane are y_0, y_2.

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so the zeroes of $y \mapsto y - x + 2tu_0(y)$ in the upper half plane are y_0, y_2. Furthermore, $y_0 + y_1 + y_2 = x$.

The solution of the resolvent equation is rational !!

For $x \in \mathbb{C}_+$ with $\mathrm{Im} x > 0$ small, we calculate $f_{t,x} := (G + 2tT_{u_0} - x)^{-1}\Pi u_0$. Notice that, for every $f \in L^2_+(\mathbb{R})$,

$$T_{u_0} f(y) = u_0(y)f(y) - \frac{\overline{c}f(i)}{y-i} \ , \ Gf(y) = yf(y) + \frac{1}{2i\pi} I_+(f) \ .$$

Therefore the resolvent equation reads

$$(y - x + 2tu_0(y))f_{t,x}(y) = u_0(y) + \lambda(t,x) + \frac{\mu(t,x)}{y-i} \ ,$$

where $\lambda(t,x), \mu(t,x)$ are chosen so that the solution is holomorphic in the upper half plane, namely so that the right hand side cancels at the zeroes $y_0(t,x), y_2(t,x)$ of $y - x + 2tu_0(y)$ in the upper half plane.

$$\begin{cases} u_0(y_0(t,x)) & +\lambda(t,x) + \frac{\mu(t,x)}{y_0(t,x)-i} = 0 \\ u_0(y_2(t,x)) & +\lambda(t,x) + \frac{\mu(t,x)}{y_2(t,x)-i} = 0 \end{cases}$$

Conclusion

Recall that $2tu_0(y_k(t,x)) = x - y_k(t,x)$, $k = 0, 1, 2$.
Solving the linear system, we get

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Since $\Pi u(t,x) = \frac{1}{2i\pi} I_+(f_{t,x}) = -\lim_{y\to\infty} y f_{t,x}(y) = -\lambda(t,x)$, we conclude that, for $x \in \mathbb{R}$,

$$\begin{aligned} u(t,x) &= -\lambda(t,x) - \overline{\lambda(t,x)} \\ &= \frac{2x - 2y_0(t,x) - y_2(t,x) - \overline{y}_2(t,x)}{2t} \\ &= \frac{x - y_0(t,x) + y_1(t,x) - \overline{y}_2(t,x)}{2t} . \end{aligned}$$

In case 1, y_0, $u(t,x) = u_0(y_0(t,x))$.
In case 2, $u(t,x) = u_0(y_0(t,x)) - u_0(y_1(t,x)) + u_0(y_2(t,x))$.

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  to describe the oscillations in the case of multiple characteristics, as was done by Claeys and Grava (2009) for a special solution of KdV.
- Can one prove such a general result for the zero dispersion limit of the Korteweg–de Vries equation ? Of a more general nonlinear dispersive equation ?
- Use the explicit formula for studying the long time behavior of BO solutions (soliton resolution) ?

THANK YOU FOR YOUR ATTENTION !