Turbulence in completely integrable PDEs: The Calogero-Moser derivative NLS

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Introduction

This talk deals with the **Calogero-Moser derivative NLS (CM-NLS)** given by

$$\boxed{\mathrm{i}\partial_t u + \partial_{xx} u + f(u, Du) = 0 \quad \text{on} \quad \mathbb{R}_t \times \mathbb{R}_x}$$

with some nonlinearity $f(u, Du)$ related to the **Hardy space** $L^2_+(\mathbb{R})$.

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2. PDE Features
3. Results
4. Outlook

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Based on `arXiv:2208.04105` joint with **Patrick Gérard** (Orsay).

Introduction: Genesis of Calogero-Moser Systems

We begin in 1975 with a seminal paper by **Jürgen Moser**.

ADVANCES IN MATHEMATICS 16, 197–220 (1975)

Three Integrable Hamiltonian Systems Connected with Isospectral Deformations*

J. MOSER

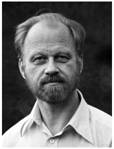
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

DEDICATED TO STAN ULAM

1. INTRODUCTION

(a) *Background.* In the early stages of classical mechanics it was the ultimate goal to integrate the differential equations of motions explicitly or by quadrature. This led to the discovery of various "integrable" systems, such as Euler's two fixed center problems, Jacobi's integration of the geodesics on a three-axial ellipsoid, S. Kovalevski's motion of the top under gravity for special ratios of the principal moments of inertia, to name a few nontrivial examples. These efforts and their climax with the work of Jacobi who applied skillfully the method of separation of variables to partial differential equations, the Hamilton–Jacobi equations associated with the mechanical system, to establish their integrable character.

However, this development took a sharp turn when Poincaré showed that most Hamiltonian systems are not integrable and gave arguments



- Moser solves a **conjecture** by F. Calogero (1971) claiming complete integrability of classical Hamiltonian systems with inverse square interaction

$$V(x) \sim \frac{1}{|x|^2}$$

- **Moser's idea:** Use **Lax pairs**!

Classical Calogero-Moser Systems

Jürgen Moser (Adv. Math. 1975) proved **complete integrability** of classical N-body system with Hamiltonian

$$\boxed{H_N = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \frac{1}{2}\sum_{k\neq\ell}^{N} \frac{1}{(x_k - x_\ell)^2}}$$

with positions $x_k \in \mathbb{R}$ and momenta $p_k \in \mathbb{R}$ for $1 \leq k \leq N$.
(Conjectured by F. Calogero (1971) based on exactly solvable QM model.)

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- Moser's recasts **equations of motion** into **Lax form:**

$$\frac{d}{dt}\mathbf{L} = [\mathbf{B}, \mathbf{L}]$$

  for suitable matrices $\mathbf{L} = \mathbf{L}(x_k, p_k)$ and $\mathbf{B} = \mathbf{B}(x_k, p_k)$ in $\mathbb{C}^{N\times N}$.
- We find N conserved quantities by traces of Moser's $\mathbf{L}$ matrix:

$$\mathrm{Tr}(\mathbf{L}^k) = \text{const.} \quad \text{for } k = 1, \ldots N.$$

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- Some extra effort shows **complete integrability** for H_N.

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$$\mathbf{M}(t) = 2\mathbf{L}_0 t + \mathbf{M}_0$$

where Moser's matrix $\mathbf{L}_0 = \mathbf{L}_{t=0}$ and $\mathbf{M}_0 = \mathrm{diag}(x_1(0), \ldots, x_N(0))$.

- The **eigenvalues** $\{x_k(t)\}_{k=1}^{N}$ **of** $\mathbf{M}(t)$ solve

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  which are the equations of motion for H_N.
- By now, many variations on this theme exist → Generalized CM-Systems.

Continuum Limit of CM-Systems

(Wiegmann et al. '09): Formal analysis of **continuum limit** of CM-Hamiltonian H_N as $N \to \infty$. the hydrodynamic fields of **mass** and **momentum density**:

$$\varrho(t,x) = \sum_{k=1}^{N} \delta(x - x_k(t)), \quad p(t,x) = \sum_{k=1}^{N} p_k(t)\delta(x - x_k(t)).$$

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- In the formal limit $N \to \infty$, they obtain that complex field $\psi = \sqrt{\varrho} e^{\mathrm{i}\vartheta}$ with $\partial_x \vartheta = p/\varrho$ solves the Hamiltonian NLS:

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- By the gauge transform $u(t,x) := \psi(t,x) e^{-\frac{\mathrm{i}}{2}\int_{-\infty}^{x} |\psi(t,y)|^2 dy}$, we arrive at

$$\boxed{\mathrm{i}\partial_t u + \partial_{xx} u + 2\Pi_+(D|u|^2)u = 0} \qquad \text{(CM-NLS)}$$

Here $D = -\mathrm{i}\partial_x$ and $\Pi_+ : L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$ is **Cauchy-Szegő projection**:

$$\widehat{(\Pi_+ f)}(\xi) = \mathbb{1}_{\xi \geq 0} \widehat{f}(\xi).$$

- Positive frequency condition interpreted as **chirality**.

Introducing the (CM-NLS)

With the short-hand notation $D_+ := \Pi_+ D$, we compactly write

$$\boxed{\mathrm{i}\partial_t u + \partial_{xx} u + 2(D_+|u|^2)u = 0} \qquad \text{(CM-NLS)}$$

- For $s \geq 0$, the Hardy-Sobolev spaces

$$H^s_+(\mathbb{R}) = L^2_+(\mathbb{R}) \cap H^s(\mathbb{R}) = \{f \in H^s(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [0, \infty)\}.$$

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- Note that the **Benjamin-Ono equation** can written in similar form:

$$i\partial_t u + \partial_{xx} u + D(u^2) + 2D_+(|u|^2) = 0 \qquad \text{(BO)}$$

Thus (CM-NLS) can be seen as L^2**-mass critical sibling** of (BO).

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- There is also a 'defocusing' version of (CM-NLS) with

$$\mathrm{i}\partial_t u + \partial_{xx} u - 2(D_+|u|^2)u = 0. \qquad \text{(INLS)}$$

  found by (Pelinovsky/Grimshaw '95) in fluid dynamics. No finite-energy solitons.

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Features of Calogero-Moser DNLS

$$\boxed{\mathrm{i}\partial_t u = -\partial_{xx} u - 2D_+(|u|^2)u}$$

- **Hamiltonian PDE** on spaces H^s_+ with energy

$$E(u) = \int_{\mathbb{R}} |Du|^2 + \ldots = \int_{\mathbb{R}} |Du - \mathrm{i}\Pi_+(|u|^2)u|^2 \quad \text{(Energy)}$$

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- Conservation of L^2-**Mass** and **Momentum**:

$$M(u) = \int_{\mathbb{R}} |u|^2 \ \ (L^2\text{-Mass}), \quad P(u) = \int_{\mathbb{R}} (\overline{u}Du - \frac{1}{2}|u|^4) \ \ \text{(Momentum)}$$

- We find **infinite set of conserved quantities** $\{I_k(u)\}_{k=0}^{\infty}$ by **Lax structure**.

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- We find **infinite set of conserved quantities** $\{I_k(u)\}_{k=0}^{\infty}$ by **Lax structure**.
- Scaling of (CM-NLS) is L^2-**critical** with

$$u(t,x) \mapsto u_\lambda = \lambda^{1/2} u(\lambda^2 t, \lambda x), \quad M(u) = M(u_\lambda)$$

Large data initial $M(u_0) \gg 1$ may not be controlled a-priori. Blowup?

Ground State Solitons

$$\boxed{i\partial_t u = -\partial_{xx} u - 2D_+(|u|^2)u}$$

- **Ground states** = Minimizers of $E(u)$. Solve **1st order equation:**

$$\boxed{\partial_x R - i\Pi_+(|R|^2)R = 0}$$

- Explicit solutions given by

$$R(x) = \frac{\sqrt{2}}{x+i} \in H^1_+(\mathbb{R}), \quad M(R) = \int_{\mathbb{R}} |R|^2 = 2\pi, \quad E(R) = 0,$$

  and $u(t,x) \equiv R(x)$ are static solutions of (CM-NLS).

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- We'll mainly focus on the large data with $M(u_0) \geq M(R)$.

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- **Ground states** $=$ Minimizers of $E(u)$. Solve **1st order equation:**

$$\boxed{\partial_x R - i\Pi_+(|R|^2)R = 0 \quad \Longleftrightarrow \quad |D|w = e^w}$$

  Equivalence to nonlocal Liouville equation in $\mathbb{R}$ via $w = \log(|R|^2)$.
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$$i\partial_t u + \partial_{xx} u + 2(D_+|u|^2)u = 0, \quad u|_{t=0} = u_0 \in H^s_+(\mathbb{R}).$$

- **Local Well-Posedness** in $H^s_+(\mathbb{R})$ for any $s > 1/2$.
- Use Kato's scheme and arguments by (de Moura-Pilod '10) for 'defocusing' version of (CM-DNLS); Tao's gauge trick for (BO).

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- **Global Well-Posedness** for subcritical L^2-mass $M(u_0) < M(R)$ with a-priori bounds

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^k} \lesssim \|u_0\|_{H^k} \quad \text{for any } k \in \mathbb{N}.$$

by **hierarchy of conserved quantities** $\{I_k(u)\}_{k=0}^{\infty}$ from Lax structure. Scattering.

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  by **hierarchy of conserved quantities** $\{I_k(u)\}_{k=0}^{\infty}$ from Lax structure. Scattering.
- **Question:** What happens for large L^2-mass $M(u_0) \geq M(R)$?

About critical L^2-mass

Theorem (GWP for Critical Mass)

Let $u_0 \in H^1_+(\mathbb{R})$ *with* **critical** L^2**-mass:**

$$M(u_0) = M(R).$$

Then the solution $u \in C(\mathbb{R}; H^1_+(\mathbb{R}))$ *of (CM-NLS) exists* **globally in time**.

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- Indeed, if $\lim_{t \to T} \|u(t)\|_{H^1} = \infty$ for some $T < \infty$, then adapting (Merle '93) for L^2-critical NLS, we deduce (up to symmetries) that initial datum is **pseudo-conformal transform of ground state**:

$$u_0(x) = e^{\mathrm{i}x^2/4T} R(x).$$

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- But by slow decay $|R(x)| \sim \frac{1}{|x|}$, we deduce that

$$\|\nabla u_0\|_{L^2} \sim \int_{\mathbb{R}} |x|^2 |R(x)|^2 = +\infty.$$

Contradiction to $u_0 \in H^1(\mathbb{R})$.

Idea (Pole Ansatz): We seek special solutions of (CM-NLS) of the form

$$\boxed{u(t,x) = \sum_{k=1}^{N} \frac{a_k(t)}{x - z_k(t)} \in H^1_+(\mathbb{R})}$$

Here $a_1(t), \ldots, a_N(t) \in \mathbb{C}$ and pairwise distinct $z_1(t), \ldots, z_N(t) \in \mathbb{C}_-$.

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- Quantization of L^2-mass: $\|u\|_{L^2}^2 = 2\pi N$.

Multi-Solitons (Naive Approach)

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- Set of nonlinear constraints for $\{a_k(t), z_k(t)\}$.
- Quantization of L^2-mass: $\|u\|_{L^2}^2 = 2\pi N$.
- Pole dynamics obey **complex Calogero-Moser system**:

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Multi-Solitons (Naive Approach)

Idea (Pole Ansatz): We seek special solutions of (CM-NLS) of the form

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- Way out: Develop **robust approach** using the Lax structure. Next!

Digression: Lax Structure

- Observe that the energy $E(u)$ can be written as a complete square

$$E(u) = \int_{\mathbb{R}} |Du - \Pi_+(|u|^2)u|^2 = \langle L_u u, L_u u \rangle$$

with the **self-adjoint, first-order and nonlocal** operator

$$\boxed{L_u f = Df - \Pi_+(u\Pi_+(\bar{u}f)) = Df - T_u T_{\bar{u}} f}$$

where $T_\varphi f := \Pi_+(\varphi f)$ on $L^2_+(\mathbb{R})$ is **Toeplitz operator** with symbol φ.

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- Spectrum $\sigma(L_{u(t)})$ is constant in time and **hierarchy of conservation laws**:

$$\boxed{I_k(u) = \langle u, L_u^k u \rangle = I(u_0) \quad \text{for } k = 0, 1, 2, \ldots}$$

Note that $M(u) = I_0(u), P(u) = I_1(u)$, and $E(u) = I_2(u)$.

Spectral Analysis of L_u

Lax Operator for (CM-NLS)

For given $u \in H^s_+(\mathbb{R})$ with $s \geq 0$, we have the self-adjoint Lax operator

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- Fun fact: $L_u^{(BO)} = D - T_u$ is Lax operator for Benjamin-Ono equation.

Multi-Solitons via Lax Structure

For each $N \in \mathbb{N}$, the Lax operator L_u admits **multi-soliton potentials**

$$u(x) = \frac{P(x)}{Q(x)} \in H^1_+(\mathbb{R}) \qquad \text{(rational function)}$$

with $\deg P = N$ and $\deg Q \leq N - 1$ and $P\overline{P} = \mathrm{i}(Q'\overline{Q} - \overline{Q}'Q)$.

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- If $Q(x)$ can have **non-simple zeros** (corresponding to **pole collisions**).

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- Develop inverse spectral formula to study dynamics of multi-solitons.

Multi-Solitons: Inverse Spectral Formula

Lemma

For any $N \geq 1$*, a multi-soliton* $u(t) \in H^1_+(\mathbb{R})$ *can be expressed as*

$$u(t,x) = \left\langle X, (\mathbf{M}(t) - x)^{-1} Y \right\rangle_{\mathbb{C}^N} \quad \textit{with} \quad \operatorname{Im} x > 0$$

with some constant vectors $X, Y \in \mathbb{C}^N$ *(depending on initial datum* u_0*).*

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- Long-time behavior of $u(t)$ by matrix perturbation analysis

$$\mathbf{M}(t) = t \left\{ 2\mathbf{L}_0 + \frac{1}{t}\mathbf{M}_0 \right\}$$

  with small parameter $\varepsilon = \frac{1}{t} \ll 1$. We shall need up to $\mathcal{O}(\varepsilon^4)$.

Turbulence of N-Solitons

Theorem (Gérard-L. '22)

For all $N \geq 2$, every multi-soliton $u(t)$ for (CM-DNLS) satisfies:

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  with some $a_\infty \in \mathbb{C}$, $z_\infty \in \mathbb{C}_-$, and velocities $v_k \neq v_l$ for $k \neq l$.
- **Striking contrast** to other integrable PDEs on the line (BO), (KdV), or the L^2-critical derivative NLS

$$\mathrm{i}\partial_t q + \partial_{xx} q + D(|q|^2 q) = 0 \qquad \text{(dNLS)}$$

  Globally well-posed in $L^2(\mathbb{R})$ with **a-priori bounds** $\|q(t)\|_{H^s} \lesssim \|q(0)\|_{H^s}$ for $s \geq 0$; see (Killip-Visan et al. '22) and (Bahouri/Perelman '20).

Outlook

Sneak Preview

(Gérard '23) found explicit solution formula for **Benjamin-Ono equation**; see talk tomorrow. Same ideas apply to (CM-NLS) to get **solution formula:**

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Thank you for your attention!