Area Variations under pointwise Lagrangian and Legendrian Constraints

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Partial Differential Equations

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resp.

$$E(u,g) := \frac{1}{2} \int_\Sigma |du|_g^2 \; dvol_g \geq A(u)$$

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(opposite of holomorphic 2-planes where iL=L)

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$\beta \in \mathbb{R}/2\pi\mathbb{Z}$ is called the Lagrangian angle

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this gives

$$d(\iota_X \omega) = d(X_1\, dy_2 - X_2\, dy_1 + X_3\, dy_4 - X_4\, dy_3) = \lambda\, \omega$$

In closed Kähler manifold where ω is non exact $\Longrightarrow$ $d(\iota_X \omega) = 0$

When $\iota_X \omega$ is exact: Hamiltonian Variations

$$\iota_X \omega = dh \quad \Leftrightarrow \quad X = J_0 \nabla h$$

When $\iota_X \omega$ is just closed: Lagrangian Variations

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Lagrangian variational problems are much more rigid than hamiltonian variational problems.

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Precisely g has isolated singularities

$$\operatorname{curl}(g^{-1}\, \nabla g) = 2\pi\, i \sum_{j=1}^{Q} d_j\, \delta_{p_j} \quad \Longrightarrow \quad g \notin W^{1,2}_{loc}(\mathbb{C}, \mathbb{C}^2)$$

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- We couldn't find a weak formulation of the E.L. equation compatible with the Lagrangian.

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We aim at study the area variations under Lagrangian contraint.

In particular perform minmax operations.

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- Can one look for non zero Morse Index Solutions without working with a PDE in a Function Space ?
- This is what Geometric Measure Theory is doing in some sense (Allard/Almgren Varifold theory).

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$$\Sigma_\varepsilon = \{(y_1, y_2, y_3, y_4) \quad ; \quad y_1^2 + y_2^2 = \varepsilon^2 \quad -1 \leq y_3 \leq +1 \quad y_4 = 0\}$$

solves the Lagrangian Plateau Problem for it's boundary.

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$$\alpha \wedge d\alpha \wedge d\alpha \quad \Longrightarrow \quad \text{Ker}(d\alpha) \text{ is non integrable}$$

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Hamiltonian variations

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where legendrian stationary is defined by

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for $h = \varphi$ and X is Hamiltonian

while

$$\pi_* X = y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3} + y_4\partial_{y_4} \neq -J_0\nabla^H h$$

is not Hamiltonian in $\mathbb{C}^2$.

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$$\sigma := \frac{2\,\varphi}{\rho^2}$$

is scaling invariant.

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Recall the Euclidian case (i.e. u minimal)

$$\theta_0+\int_{\rho<r}\frac{|(\nabla^\Sigma\rho)^\perp|^2}{\rho^2}\ dvol_{g_u} = \frac{1}{r^2}\int_{\rho<r} dvol_{g_u}$$

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Too few variations for hoping a regularity theory.

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A Digression to "Target Harmonic Maps"

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$$\int_\Omega \nabla u \cdot \nabla[X(u)]\ dx^2 = 0$$

for a.e. Ω and every $X \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$ s.t.

$$u(\partial\Omega) \cap \mathrm{Supp}(X) = \emptyset\ .$$

Theorem [R. 2017] $W^{1,2}$ Target harmonic map are conformal harmonic in a classical sense and hence C^∞.

Parametrized Stationary Integer Varifolds

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Theorem [R. Pub IHES 2017] Every non trivial minmax operation for the area of surfaces is realized by a *PSIV*

Theorem [Pigati, R. Duke 2020] Every *PSIV* is a smooth branched immersion equipped with a smooth multiplicity Q

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Theorem [R. 2023] Every non trivial minmax operation for the area of surfaces within Legendrian maps is realized by a PLSIV and $u \in C^0(\Sigma, N^5)$.

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Conjecture : Every *Parametrized <u>Legendrian</u> Stationary Integer Varifolds* is a smooth branched immersion away from isolated *Schoen-Wolfson cones* and equipped with a smooth multiplicity Q

- $$u \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{R}^5) \quad , \quad Q \in L^\infty(\mathbb{C}, \mathbb{N}^*) \ ,$$
- $$u^* \left(-d\varphi + y_1\, dy_2 - y_2\, dy_1 + y_3\, dy_4 - y_4\, dy_3\right) = 0 \ ,$$

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