On continuous time bubbling for the harmonic map heat flow in two dimensions

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Harmonic map heat flow

Gradient flow of the Dirichlet energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u(x)|^2 \,\mathrm{d}x,$$
$$u : \mathbb{R}^2 \to \mathbb{S}^2$$

solves the heat equation (Eells, Sampson '64):

$$u_t = \Delta u + |\nabla u|^2 u = \mathcal{T}(u)$$
$$u(0,\cdot) = u_0(\cdot)$$

Tension: $\mathcal{T}(u) = \Pi_{T_u}\Delta u$ projection onto the tangent plane T_u
Energy monotone:

$$E(u(0)) - E(u(t)) = \int_0^t \|\partial_s(s,\cdot)\|_2^2 \,\mathrm{d}s$$

Existence, regularity, energy concentration and singularities in finite time: (Struwe '85). Harmonic maps are stationary solutions to HMHF.

Struwe's heat flow

Let $\mathcal{M}, \mathcal{N}$ be general Riemannian manifolds, $\dim M = 2$.

Theorem (Struwe '85)

Initial data $u_0 \in \dot{H}^1(\mathcal{M};\mathcal{N})$, there exists unique global HMHF energy evolution on $[0,\infty) \times \mathbb{S}^2$ which is smooth up to finitely many points (x_ℓ, T_ℓ) characterized by the condition

$$\limsup_{t \to T_\ell -} E_R(u(t,\cdot), x_\ell) > \varepsilon_0 > 0$$

for all $0 < R \leq R_0$.

Local compactness in $\dot{H}^2(\mathcal{M};\mathcal{N})$ if energy does not concentrate, and $\int_P |\nabla u|^4 \, dtdx < \infty$ where P is a parabolic cylinder.

Energy concentration the only obstruction to local $\dot{H}^2$ compactness of a Palais-Smale sequence relative to energy and its L^2-gradient.

Harmonic sphere bubbles off at any singular time.

Chang, Ding, Ye '92: Finite time blowup.

Qing's bubbling theorem

Jie Qing '95 characterized singularity formation in Struwe's HMHF $\mathbb{R}^2 \to \mathbb{S}^2$ via a bubble decomposition along a carefully chosen sequence of times approaching one of the singular times T_ℓ.

Theorem (Qing '95)

Let (x_0, T_0) *be a singularity of* $u : [0, \infty) \times \mathbb{R}^2 \to \mathbb{S}^2$*, HMHF solution. There exist* $t_n \to T_0-$*, harmonic spheres* $\omega_k : \mathbb{R}^2 \to \mathbb{S}^2$

$$\lim_{t \to T_0-} E_R(u(t, \cdot), x_0) = E_R(u(T_0, \cdot), x_0) + \sum_{k=1}^{p} E(\omega_k)$$

$$u(t_n, \cdot) = u(T_0, \cdot) + \sum_{k=1}^{p} \Big(\omega_k \big(\frac{\cdot - a_n^k}{\lambda_n^k} \big) - \omega_k(\infty) \Big) + o_{W^{1,2}(B_R)}(1)$$

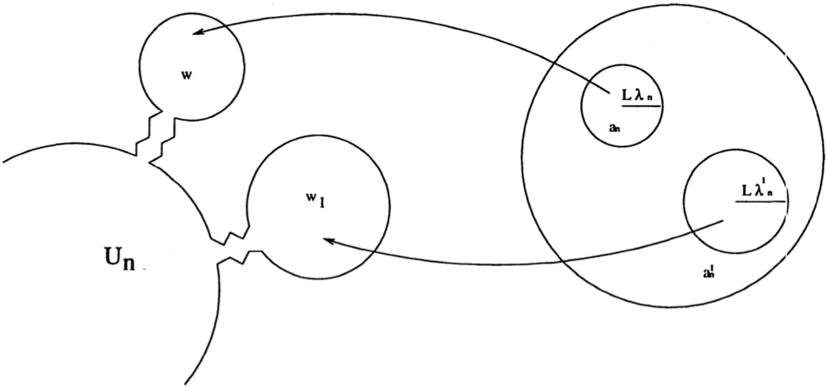
$R > 0$ *small,* $\lambda_n^k \to 0$*,* $a_n^k \to x_0$*. Bubbles* asymptotically orthogonal.

Proved via bubbling for a Palais-Smale sequence.

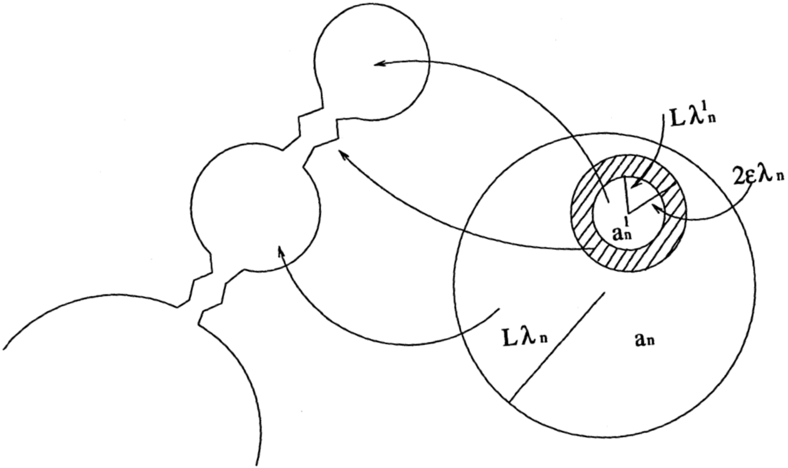
Asymptotic orthogonality of the bubbles

For all $k \neq \ell$, $n \to \infty$

$$\frac{\lambda_n^k}{\lambda_n^\ell} + \frac{\lambda_n^\ell}{\lambda_n^k} + \frac{|a_n^k - a_n^\ell|^2}{\lambda_n^k \lambda_n^\ell} \to \infty \tag{1}$$



Bubbles on bubbles (from Qing's paper)



Harmonic maps

Theorem

$u : \mathbb{R}^2 \to \mathbb{S}^2$ weak non-constant solution of $\Delta u + u|\nabla u|^2 = 0$ of finite energy. Then $u : \mathbb{S}^2 \to \mathbb{S}^2$ smooth harmonic map (Hélein, Sacks-Uhlenbeck), nonzero degree. Conformal modulo orientation (Eells-Wood). Cauchy-Riemann system

$$\partial_1 u \mp u \times \partial_2 u = 0 \iff \partial_2 u \pm u \times \partial_1 u = 0$$

holds, u unique minimizer of energy in its homotopy class, $E(u) = 4\pi|\deg(u)|$. There exist $P, Q \in \mathbb{C}[z]$ without common linear factor satisfying

$$\max(\deg(P), \deg(Q)) = |\deg(u)| \geq 1$$

and such that $u = \frac{P}{Q}$ for $deg(u) > 0$, or $\bar{u} = \frac{P}{Q}$ for $\deg(u) < 0$.

Key steps in the proof

- Hélein's regularity theorem (false in $\mathbb{R}^d, d \geq 3$). Div, curl structure, Hardy space compensated compactness (Coifman, Lions, Meyer, Semmes '92): continuity of weak solution. Then by elliptic regularity $\nabla u \in L^p$, $u \in C^\infty(\mathbb{R}^2)$
- Hopf quadratic differential

$$\varphi \, dz^2 = \langle \partial_z u, \partial_z u \rangle \, dz^2 = \left(|\partial_x u|^2 - |\partial_y u|^2 - 2i\langle u_x, u_y \rangle \right) dz^2$$

  Harmonic map: $\partial_{\bar{z}} \varphi = 0$ holomorphic on $\mathbb{S}^2$, constant. Vanishes at $z = \infty$ so conformality follows:

$$|\partial_x u|^2 - |\partial_y u|^2 - 2i\langle u_x, u_y \rangle = 0$$

- Bogomolnyi identity:

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_1 u - u \times \partial_2 u|^2 + \int_{\mathbb{R}^2} \partial_1 u \cdot u \times \partial_2 u$$

Elliptic compactness lemma: bubbling in energy and L^∞

$u_n : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ with $\limsup_{n\to\infty} E(u_n) < \infty$, and

$$\lim_{n\to\infty} \rho_n \|\mathcal{T}(u_n)\|_{L^2} = 0$$

for some $\rho_n \in (0,\infty)$. For arbitrary $y_n \in \mathbb{R}^2$, $\exists\ R_n \to \infty$ with

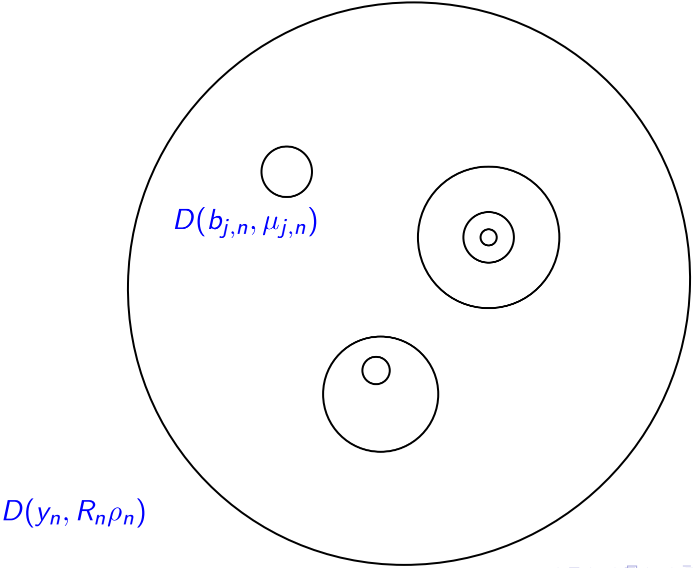
$$u_n - \omega_0\Big(\frac{\cdot - y_n}{\rho_n}\Big) - \sum_{j=1}^{M} \Big(\omega_j\Big(\frac{\cdot - b_{j,n}}{\mu_{j,n}}\Big) - \omega_j(\infty)\Big) \to 0$$

in energy and uniformly on $D(y_n, R_n\rho_n) \supset D(b_{j,n}, \mu_{j,n})$

- harmonic maps ω_j, nonconstant if $j \geq 1$
- orthogonality of scales as in (1)
- separation of $D(b_{j,n}, \mu_{j,n})$ from $\partial D(y_n, R_n\rho_n)$
- quantization of energy: $E(u_n; D(y_n, R_n\rho_n)) = 4\pi K + o(1)$

Qing '95, Ding-Tian '95, Wang '96, Qing-Tian '97, Lin-Wang '98

Disks in the bubble tree



Local Palais-Smale sequences for the heat flow

Smooth HMHF $u : [0, T) \times \mathbb{S}^2 \to \mathbb{S}^2$, singularity at $t = T$. Energy dissipation

$$\int_0^T \|\mathcal{T}(u)(t)\|_2^2 \,\mathrm{d}t < \infty \tag{2}$$

- If $T = \infty$, then $\exists\ t_n \to \infty$ with $\sqrt{t_n}\|\mathcal{T}(u(t_n))\|_2 \to 0$
- If $T < \infty$, then $\exists\ t_n \to T-$ with $\sqrt{T - t_n}\|\mathcal{T}(u(t_n))\|_2 \to 0$

Elliptic compactness applies at these parabolic scales. Rescale

- If $T = \infty$, then $u_n(y) := u(t_n, y_n + \sqrt{t_n}\, y)$ is Palais-Smale
- If $T < \infty$, then $u_n(y) := u(t_n, y_n + \sqrt{T - t_n}\, y)$ is Palais-Smale

Bubbling for HMHF locally at parabolic scales along a time sequence t_n determined by L^2 integrability (2).

Open problems

- If $T < \infty$, is the body map $u(T, \cdot)$ continuous?
- If $T = \infty$, are the points of energy concentration unique?
- Uniqueness of harmonic bubbles? Counterexamples by Topping if target manifold not $\mathbb{S}^2$ (nonanalytic)
- Continuous in time bubbling (soliton resolution)?

Progress by Topping, '97, '04 for maps $\mathbb{S}^2 \to \mathbb{S}^2$.

Theorem (Topping, '97, '04)

If $T = \infty$ and if all the concentrating bubbles in the sequential decomposition have the same orientation, then the points of energy concentration $\{x_\ell\} \subset \mathbb{S}^2$ are unique. Moreover, the body map is unique, i.e., there exists a harmonic map $\omega_\infty : \mathbb{S}^2 \to \mathbb{S}^2$ such that $u(t) \rightharpoonup \omega_\infty$ as $t \to \infty$, weakly in $\dot{H}^1$ and strongly in $C^k_{loc}(\mathbb{S}^2 \setminus \{x_\ell\})$.

Soliton resolution in the equivariant case

Consider k-equivariant maps $u : \mathbb{R}^2 \to \mathbb{S}^2$, i.e.,

$$u(t, re^{i\theta}) = (\sin\psi(t,r)\cos k\theta, \sin\psi(t,r)\sin k\theta, \cos\psi(t,r))$$

Harmonic maps given by $\psi(t,r) = m\pi \pm Q(r/\lambda)$ for $m \in \mathbb{Z}$, $\lambda > 0$, and $Q(r) = 2\arctan(r^k)$.

Theorem (Jendrej-Lawrie '22)

Let $\psi(t,r)$ solve the HMHF. Suppose $T = \infty$. Then, there exist $m \in \mathbb{Z}$, $N \in \mathbb{N}$ and C^1 functions $0 < \lambda_1(t) < \cdots < \lambda_N(t)$ such that,

$$\lim_{t\to T} \|\psi(t,\cdot) - m\pi - \sum_{j=1}^{N} \pm(Q(\cdot/\lambda_j(t)) - \pi)\|_{\mathcal{E}} = 0$$

and $\lim_{t\to T} \sum \lambda_j(t)/\lambda_{j+1}(t) = 0$. Similar when $T < \infty$.

Note: $\lambda_{N+1}(t) := \sqrt{t}$, and subsequent equivariant bubbles always have opposite orientations as maps $\mathbb{R}^2 \to \mathbb{S}^2$.

Comments

- Van der Hout ('03): same result in the case $T < \infty$ by showing there are no non-trivial equivariant bubble towers in finite time. In the case $T = \infty$, non-trivial bubble towers can occur; see for example Del Pino, Musso, Wei ('21) for a construction for the closely related energy critical heat equation.
- Finite time blow up solutions with one bubble (including a stable regime) were discovered by Raphaël-Schweyer ('13, '14) for $k = 1$. See also Guan, Gustafson, Tsai ('09) and Gustafson, Nakanishi, Tsai ('10) who proved asymptotic stability of Q for $k \geq 3$, and Davila, Del Pino, Wei ('20) for blow up outside of equivariant symmetry.
- Remainder of the talk: discuss a continuous in time bubble decomposition in the general case, i.e., for maps $\mathbb{R}^2 \to \mathbb{S}^2$ without symmetry assumptions (as in Jendrej-Lawrie '22), and without assumptions on the orientations of the bubbles (as in Topping '97, '04).

Multi-bubble configuration, centers, scales

Centers and scales of harmonic maps: $\omega : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$ positive energy, $\gamma_0 \in (0, 2\pi)$, scale of ω

$$\lambda(\omega; \gamma_0) := \inf\{\lambda \in (0,\infty) \mid \exists\, a \in \mathbb{R}^2 \text{ s.t. } E(\omega; D(a,\lambda)) \geq E(\omega) - \gamma_0\}.$$

Center of ω: fix $a = a(\omega;\gamma_0) \in \mathbb{R}^2$ with

$$E(\omega; D(a(\omega;\gamma_0), \lambda(\omega;\gamma_0))) \geq E(\omega) - \gamma_0.$$

M-bubble configuration $\Omega = (\omega_0, \omega_1, \ldots, \omega_M)$

$$\mathcal{Q}(\Omega; x) = \omega_0 + \sum_{j=1}^{M} (\omega_j(x) - \omega_j(\infty))$$

where $\omega_0 = \mathrm{const} \in \mathbb{S}^2$, $\omega_j : \mathbb{R}^2 \to \mathbb{S}^2$, $j \geq 1$ non-constant harmonic maps, $\omega_j(\infty) := \lim_{|x|\to\infty} \omega_j(x)$. Constant maps: $M = 0$.

Distance to a multi-bubble configuration

Smooth map $u : \mathbb{R}^2 \to \mathbb{S}^2$, multi-bubble $\mathcal{Q}(\Omega)$, disk $D(y;\rho) \subset \mathbb{R}^2$, auxiliary scales $\vec{\nu} = (\nu, \nu_1, \ldots, \nu_M)$, $\vec{\xi} = (\xi, \xi_1, \ldots, \xi_M)$.

Distance $\mathbf{d}(u, \mathcal{Q}(\Omega); D(y,\rho); \vec{\nu}, \vec{\xi}) \ll 1$ means

- closeness in energy to multi-bubble on the large disk: $E\left(u - \mathcal{Q}(\Omega); D(y,\rho)\right) \ll 1$
- near constancy on the exterior neck region: $E(u; D(y,\nu) \setminus D(y,\xi)) + \|u - \omega_0\|_{L^\infty(D(y,\nu)\setminus D(y,\xi))} \ll 1$
- large exterior neck: $\xi \ll \rho \ll \nu$
- orthogonality of bubbles scales/centers: $\lambda(\omega_j) \ll \lambda(\omega_k)$ or $\lambda(\omega_j) \gg \lambda(\omega_k)$ or $|a(\omega_j) - a(\omega_k)| \gg \lambda(\omega_j)$
- separation from exterior neck: $\xi_j \ll \lambda(\omega_j) \ll \mathrm{dist}(a(\omega_j), \partial D(y,\xi))$

L^∞ control on the bubbles, removal of sub-bubbles

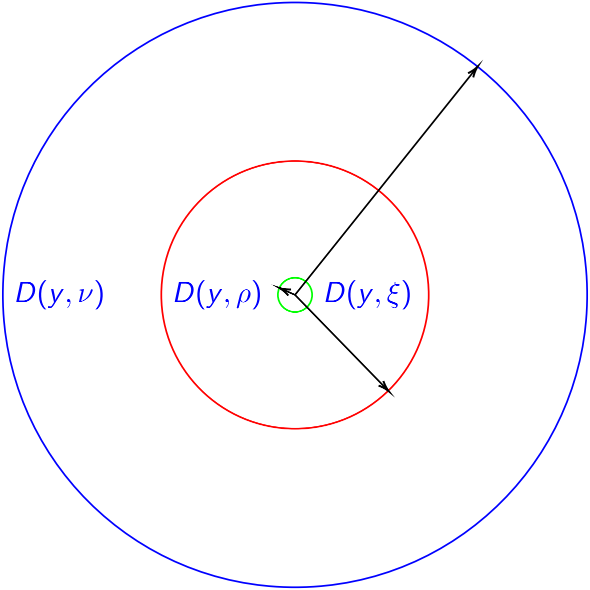
- uniform closeness of u, ω_j after removal of interior bubbles: $\|u - \omega_j\|_{L^\infty(D_j^*)} \ll 1$
- Swiss cheese (holes are of the same size): $D_j^* := D(a(\omega_j), \nu_j) \setminus \bigcup_k' D(a(\omega_k), \xi_j)$.
- separation from boundaries: $\xi_j \ll \mathrm{dist}(a(\omega_k), \partial D(a(\omega_j), \nu_j)), \quad \lambda(\omega_j) \ll \nu_j$

Local multi-bubble proximity function:

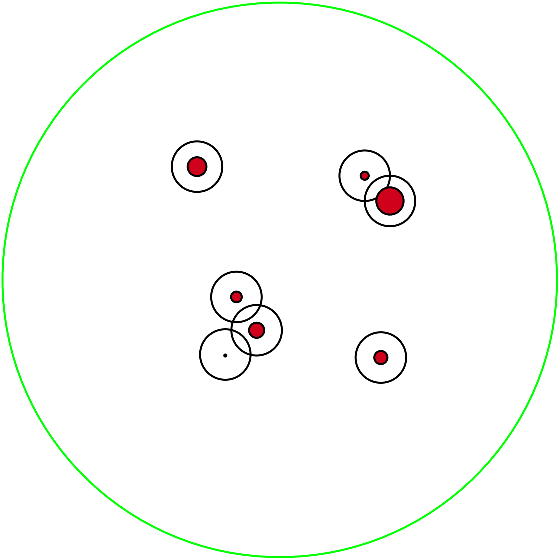
$$\boldsymbol{\delta}(u; D(y,\rho)) := \inf_{\Omega, \vec{\nu}, \vec{\xi}} \mathbf{d}(u, \mathcal{Q}(\Omega); D(y,\rho); \vec{\nu}, \vec{\xi})$$

Infimum taken over all multi-bubble configurations, and scales $\vec{\nu}, \vec{\xi}$

Exterior neck region



Swiss cheese structure



Continuous time bubbling

Theorem (Jendrej, Lawrie, S. '23)

$u(t) : [0, T_+) \times \mathbb{R}^2 \to \mathbb{S}^2$ smooth HMHF solution, maximal $T_+ = T_+(u_0) \in (0, \infty]$. If $T_+ < \infty$, then $\forall\, y \in \mathbb{R}^2$,

$$\lim_{t \to T_+} \boldsymbol{\delta}\big(u(t); D(y, \sqrt{T_+ - t})\big) = 0.$$

Arbitrary $t_n \to T_+$ and $D(y_n, R_n \rho_n) \subset D(y, \sqrt{T_+ - t})$, $R_n \to \infty$, assume energy evacuates from necks of disks. Then,

$$\lim_{n \to \infty} \boldsymbol{\delta}\big(u(t_n); D(y_n, \rho_n)\big) = 0.$$

Analogous statement on $D(y, \sqrt{t})$ if $T_+ = \infty$.

Solution remains close to multi-bubble configurations at parabolic scales, and on all smaller disks whose boundaries do not intersect bubbles, for all times up to T_+.

Comments on the theorem

- Analogous result when $T_+ = \infty$
- Does not give the uniqueness of bubbles.
- How to think about the theorem: non-existence of bubble collisions that destroy multi-bubble structure.
- As a corollary, we obtain a sequential bubble decomposition as in Qing along every time sequence $t_n \to T_+$ after passing to a subsequence (not just along Palais-Smale sequences)

Comments on the proof

- Proof by contradiction: $u(t)$ cannot come close to, and then move away from multi-bubble configurations (MBCs) infinitely many times. Reminiscent of invariant manifold theory in dynamical systems, theory of ω-limit sets.
- However: linearized operator here has no spectral gap, no stable/unstable manifolds
- By *sequential* soliton resolution (bubbling along sequence of times) we know that we approach MBCs infinitely many times.
- If theorem fails, $\boldsymbol{\delta}\big(u(t_n); D(y_n, \rho_n)\big) > \eta > 0$ for $t_n \to T_+-$. By energy dissipation and compactness lemma exist σ_n with $\boldsymbol{\delta}\big(u(\sigma_n); D(y_n, \rho_n)\big) \to 0$ where $0 < t_n - \sigma_n \ll \rho_n^2$
- Notions of collision intervals and minimal collision energy needed to lead this to a contradiction. This was essential for soliton resolution for wave maps by Jendrej, Lawrie '21.

Propagation estimates: local energy

Local energy propagation (Struwe '85): $0 < t_1 < t_2 < T_+$,

$$\int_{\mathbb{R}^2} |\nabla u(t_2,x)|^2 \phi(x)^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^2} |\nabla u(t_1,x)|^2 \phi(x)^2 \,\mathrm{d}x + CE(u_0)\frac{t_2 - t_1}{R^2}$$

$$\int_{\mathbb{R}^2} |\nabla u(t_2,x)|^2 \phi(x)^2 \,\mathrm{d}x \geq \int_{\mathbb{R}^2} |\nabla u(t_1,x)|^2 \phi(x)^2 \,\mathrm{d}x$$
$$-C\Big(E(u_0)\frac{(t_2 - t_1)}{R^2} + |E(u(t_1)) - E(u(t_2))|\Big)$$

ϕ cut-off adapted to $D(x_0, R)$.

- Integrate HMHF by parts against $u_t\phi^2$. Nonlinear term drops out, normal vector field.
- Controls energy flow on parabolic regions.
- Energy evacuates from boundaries of parabolic regions. No self-similar energy concentration both in finite (Topping) and infinite times.

Tao's $L^2_t L^\infty_x$ parabolic Strichartz estimate

Lemma: *Solution of* $\partial_t v - \Delta v = F$, $v(0) = v_0$ *satisfies*

$$\|v\|_{L^2(I;L^\infty(\mathbb{R}^2))} \leq C_0\big(\|v_0\|_{L^2(\mathbb{R}^2)} + \|F\|_{L^1(I;L^2(\mathbb{R}^2))}\big)$$

With $(Tf)(t) := e^{t\Delta}f$ one has $T^*F = \int_0^\infty e^{s\Delta}F(s)\,ds$. From

$$(TT^*F)(t) = \int_0^\infty e^{(t+s)\Delta}F(s)\,ds$$

conclude

$$\|(TT^*F)(t)\|_\infty \lesssim \int_0^\infty (t+s)^{-1}\|F(s)\|_1\,ds$$

$$\|TT^*F\|_{L^2((0,\infty),L^\infty(\mathbb{R}^2))} \lesssim \|F\|_{L^2((0,\infty),L^1(\mathbb{R}^2))}$$

$$\langle TT^*F, F\rangle = \|T^*F\|_2^2 \lesssim \|F\|^2_{L^2((0,\infty),L^1(\mathbb{R}^2))}$$

Propagation estimates: pointwise bounds

Lemma: *On a Swiss cheese region with $L \geq 0$ congruent, well-separated holes, assume*

$$\|u_{n,0} - \omega\|_{L^\infty(D(0,4R_n)\setminus\bigcup_{\ell=1}^L D(x_\ell,4^{-1}\varepsilon_n))}$$

$$+ E\Big(u_{n,0} - \omega; D(0,4R_n) \setminus \bigcup_{\ell=1}^{L} D(x_\ell, 4^{-1}\varepsilon_n)\Big) \to 0.$$

Then, if $\tau_n \ll \varepsilon_n^2$ (or $\tau_n \ll R_n^2$ if $L = 0$),

$$\|u_n(\tau_n) - \omega\|_{L^\infty(D(0,R_n)\setminus\bigcup_{\ell=1}^L D(x_\ell,\varepsilon_n))} \to 0.$$

- Contraction of heat flow on L^∞
- Tao's parabolic Strichartz estimate
- Struwe's small energy local $\int(|\nabla u|^4 + |\Delta u|^2)\, dtdx$ bound

Minimal collision energy

Definition: $K \geq 1$ minimal with the following properties.
$\exists\ y_n \in \mathbb{R}^2$, $\rho_n, \varepsilon_n \in (0, \infty)$, $\sigma_n, \tau_n \in (0, T_+)$ and $\eta > 0$, with $\varepsilon_n \to 0$, $0 < \sigma_n < \tau_n < T_+$, $\sigma_n, \tau_n \to T_+$, so that

1. $\boldsymbol{\delta}(u(\sigma_n); D(y_n, \rho_n)) \leq \varepsilon_n$;
2. $\boldsymbol{\delta}(u(\tau_n); D(y_n, \rho_n)) \geq \eta$;
3. the interval $I_n := [\sigma_n, \tau_n]$ satisfies $|I_n| \leq \varepsilon_n \rho_n^2$;
4. $E(u(\sigma_n); D(y_n, \rho_n)) \to 4K\pi$ as $n \to \infty$;

We call σ_n bubbling times, and τ_n ejection times.

Lemma: *If theorem fails, then $K \geq 1$ well-defined with collision intervals $[\sigma_n, t_n]$.*

Based on energy dissipation and localized sequential bubbling. For $K > 0$ need propagation estimates, both in energy and L^∞.

Lengths of collision intervals

Key Lemma: Let $K \geq 1$ minimal collision energy, $I_n := [\sigma_n, \tau_n]$ associated collision intervals. $\exists\, \varepsilon > 0$ such that if $s_n \in I_n$ satisfies

$$\boldsymbol{\delta}(u(s_n); D(y_n, \rho_n)) \leq \varepsilon$$

Then,

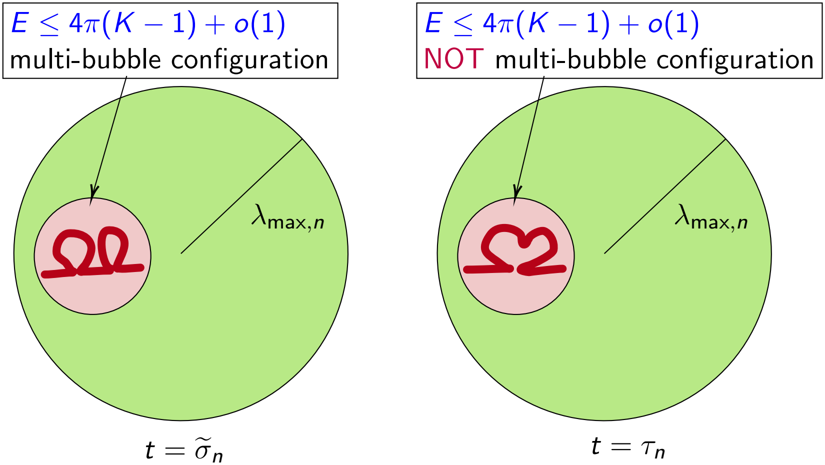
$$\tau_n - s_n \geq \varepsilon \max_{j \in \{1,\ldots,M\}} \lambda(\omega_j)^2 =: \varepsilon \lambda^2_{\max,n}. \tag{3}$$

where scales $\lambda(\omega_j)$ correspond to any MBC $\mathcal{Q}(\boldsymbol{\omega})$ for which

$$\varepsilon \leq \mathbf{d}(u(s_n), \mathcal{Q}(\boldsymbol{\omega}); D(y_n, \rho_n); \vec{\nu}, \vec{\xi}) \leq 2\varepsilon. \tag{4}$$

Proof Sketch: If lemma fails, $\exists\, \widetilde{\sigma}_n \in I_n$ with $\tau_n - \widetilde{\sigma}_n \ll \lambda^2_{\max,n}$ and for which $\boldsymbol{\delta}(u(\widetilde{\sigma}_n); D(y_n, \rho_n)) \to 0$ and $\boldsymbol{\delta}(u(\tau_n); D(y_n, \rho_n)) \geq \eta$

Key lemma: proof sketch



- By propagation estimates, multi-bubble structure is preserved at scale $\lambda_{\max,n}$ on the interval $[\widetilde{\sigma}_n, \tau_n]$.
- Hence, it is lost at a smaller scale (pink disks, radius $\sqrt{\tau_n - s_n} \ll \widetilde{\rho}_n \ll \lambda_{\max,n}$), contradicting minimality of K

Main theorem: proof sketch

- Use key lemma: fix $\varepsilon > 0$ and $J_n := [s_n, \tau_n] \subset I_n$ so that

$$\tau_n - s_n \geq \varepsilon \lambda_{\max,n}^2, \quad \delta(u(t); D(y_n, \rho_n)) \geq \varepsilon, \quad \forall t \in J_n$$

("no return property" on J_n).

- Then,

$$\lambda_{\max,n} \|\mathcal{T}(u(t))\|_2 \geq c_0 > 0 \text{ for all } t \in J_n$$

Otherwise, bubbling at scale $\lambda_{\max,n}$ at some $t_n \in J_n$ by elliptic compactness lemma, contradicting no-return property of J_n.

- Contradiction with the energy identity:

$$\infty = \sum_n \int_{s_n}^{\tau_n} c_0 \lambda_{\max,n}^{-2} \, \mathrm{d}t \leq \sum_n \int_{s_n}^{\tau_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t$$

$$\leq \int_0^{T_+} \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t < \infty$$