Recent Advances in Wave Turbulence Theory And Practice

Jalal Shatah
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June 12, 2023

SIMONS COLLABORATION ON WAVE TURBULENCE

This collaboration is a joint effort of several groups of mathematicians and physicist, with theoretical, experimental, and numerical expertise, who are working together on an interdisciplinary set of problems in Wave Turbulence Theory.

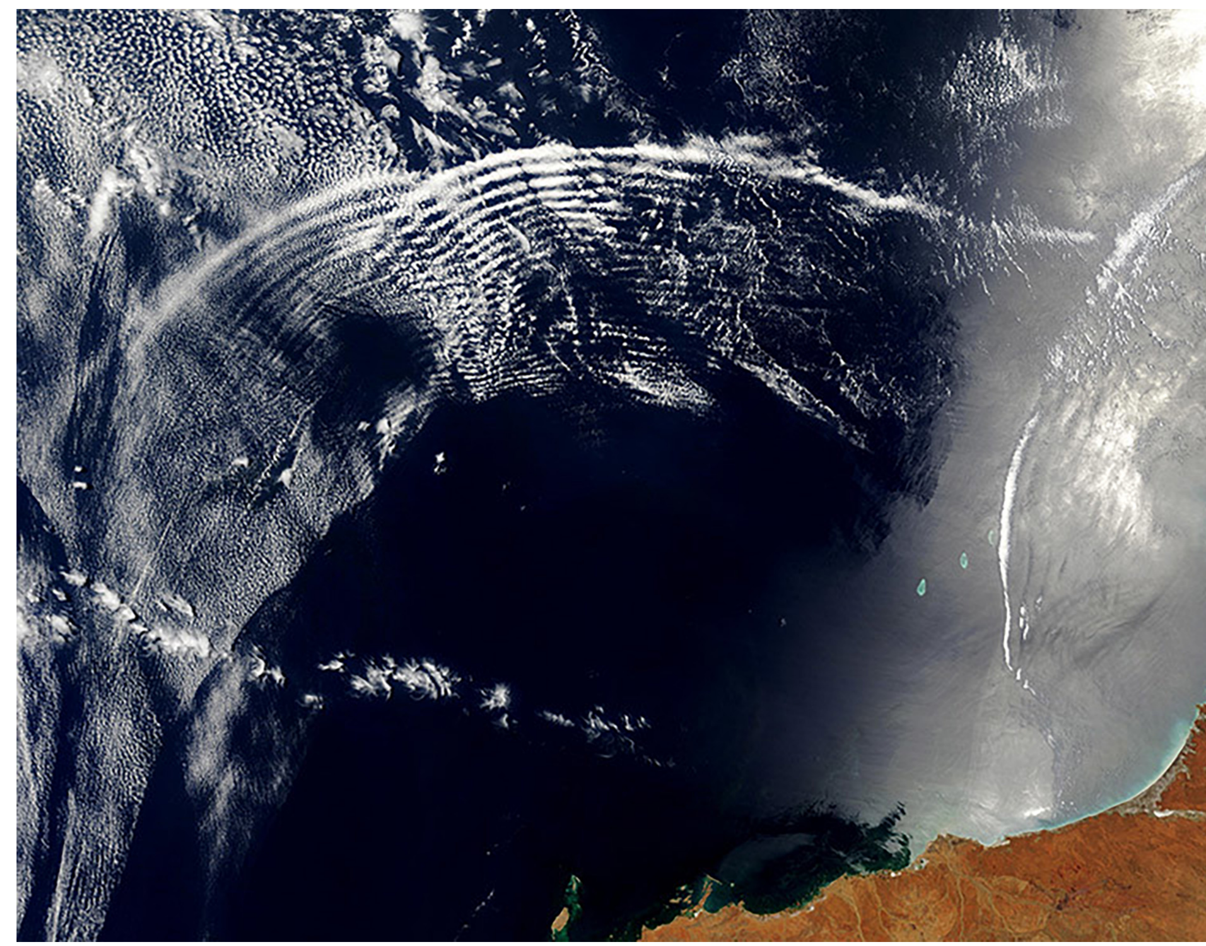
Tristan Buckmaster, Oliver Bühler, Laurent Chevillard, Pierre-Philippe Cortet, Thierry Dauxois, Eric Falcon, Erwan Faou, Isabelle Gallagher, Pierre Germain, Zaher Hani, Alexandru Ionescu, Giorgio Krstulovic, Nicolas Mordant, Andrea Nahmod, Sergey Nazarenko, Miguel Onorato, Laure Saint Raymond, Jalal Shatah, Gigliola Staffilani, Eric Vanden-Eijnden

Wave Turbulence Theory (WTT)

Waves, weakly nonlinear dispersive.

Turbulence, the presence of energy cascades between scales.

Theory of nonlinear interacting random wave (observed physically or exp.)



Wave Turbulence Theory (WTT)

$$\begin{cases} \partial_t u = i\mathcal{L}(\frac{1}{i}\partial_x)u + \mu\mathcal{N}(u), \quad x \in \mathcal{D} \\ u(x,0) = u_0(x) = \mathcal{O}(1). \end{cases}$$

Weakly nonlinear: μ *-strength of the nonlinear interactions.*

Dispersive: $\left[\partial_i \partial_j \mathcal{L}(k)\right]$*-nondegenerate*

Random waves: u_0 *-decomposed into waves, and each is randomize by an iid RV*

Wave Kinetic Equation

1) WKE was derived to describe evolution of the wave spectrum.

2) Kolmogorov–Zakharov (KZ) spectra: these are power law solutions that are analogous to the Kolmogorov spectrum of hydrodynamic turbulence describing energy cascade.

3) KZ solutions put WT into the domain of general Turbulence: strongly non-equilibrium statistical systems.

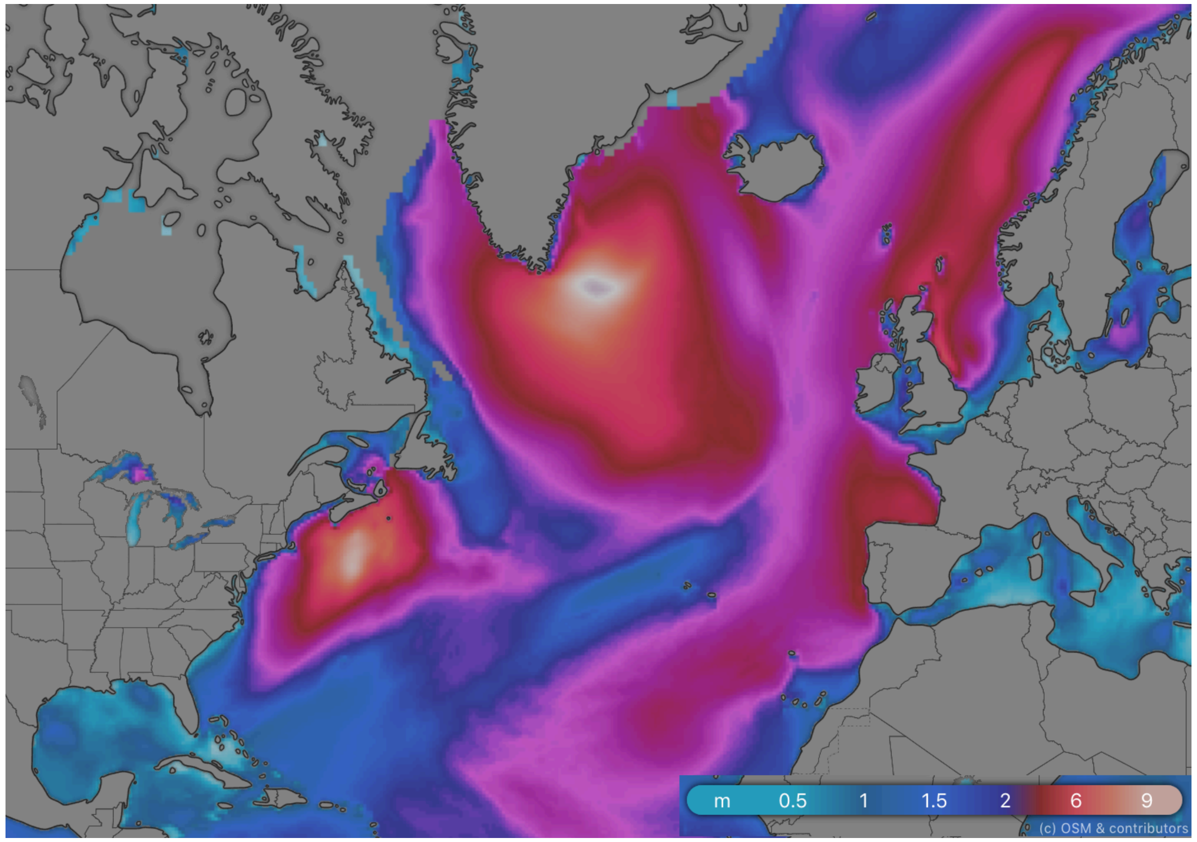


Figure: A typical map of averaged wave height obtained by integrating the WKE using wave forecasting models.

EXAMPLES

NONLINEAR SCHRÖDINGER EQUATIONS (NLS) $\omega_k \overset{def}{=} \mathcal{L}(k) = |k|^2$

$$iu_t + \Delta u \pm \mu |u|^{2N} u = 0, \; x \in \mathbb{T}_L^d$$

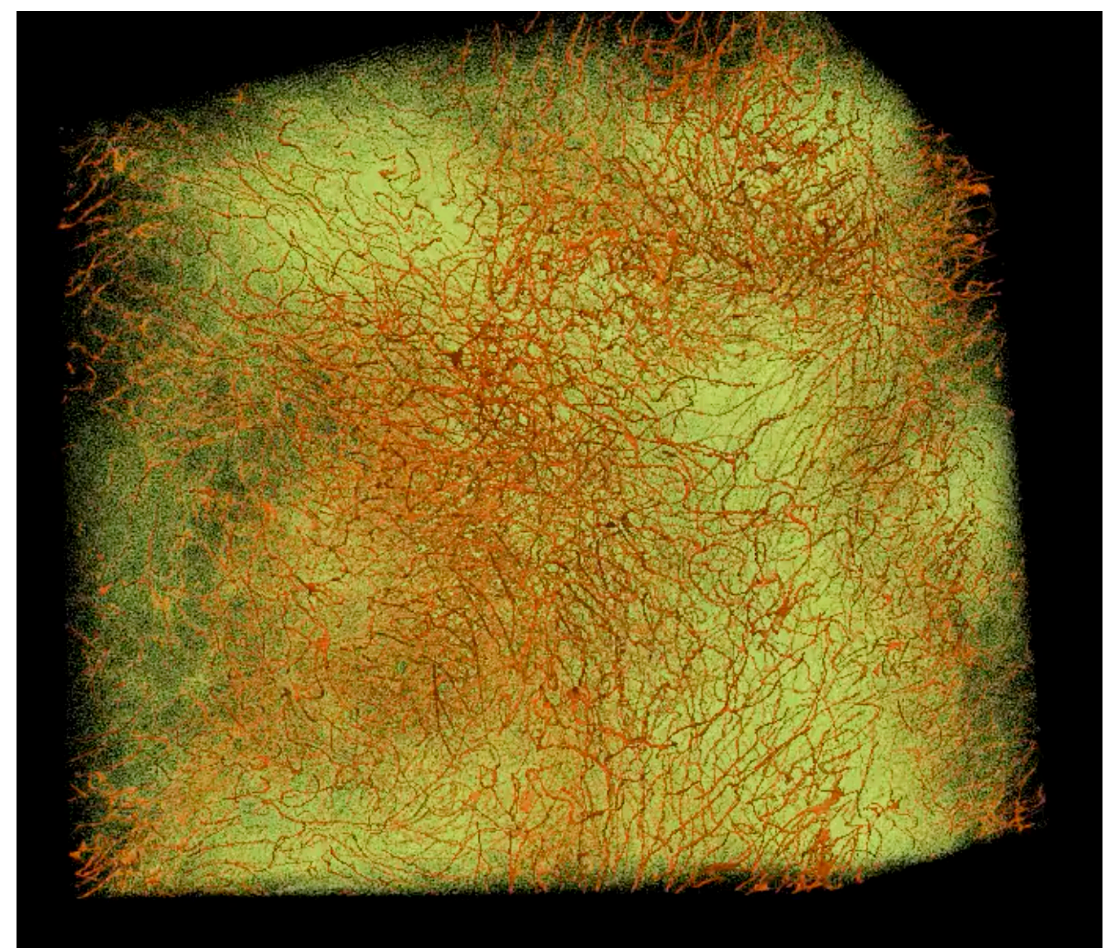
$$u(x,0) = u_0(x) = \mathcal{O}(1),$$

The Gross-Pitaevskii equation

Kelvin wave cascade & vortex filaments

Krstulovic , Müller, and Polanco

$$u = A_0 e^{-i\omega_0 t} + \delta u \qquad \omega_k = \sqrt{2A_0^2 k^2 + k^4}$$



Müller, Krstulovic
Phys. Rev. B 102, 134513 (2020)

Examples

Gravity and capillary surface waves of irrotational flow of an inviscid (ideal) incompressible and homogeneous fluid of infinite depth.

$$H = \frac{1}{L^d}\left(\sum_k \omega_k |u_k|^2 + \sum M_{34}^{12}\delta_{34}^{12} u_{k_1} u_{k_2} \bar{u}_{k_3} \bar{u}_{k_4} + \ldots\right), \quad \omega_k = |k|^{\frac{1}{2}},$$

$$H = \frac{1}{L^d}\left(\sum_k \omega_k |u_k|^2 + \sum_{S=0} M_{3}^{12}\delta_{3}^{12} u_{k_1} u_{k_2} \bar{u}_{k_3} + \ldots\right), \quad \omega_k = |k|^{\frac{3}{2}}$$

Capillary waves 1.7 cm *Wave length* 10 cm *Gravity waves*

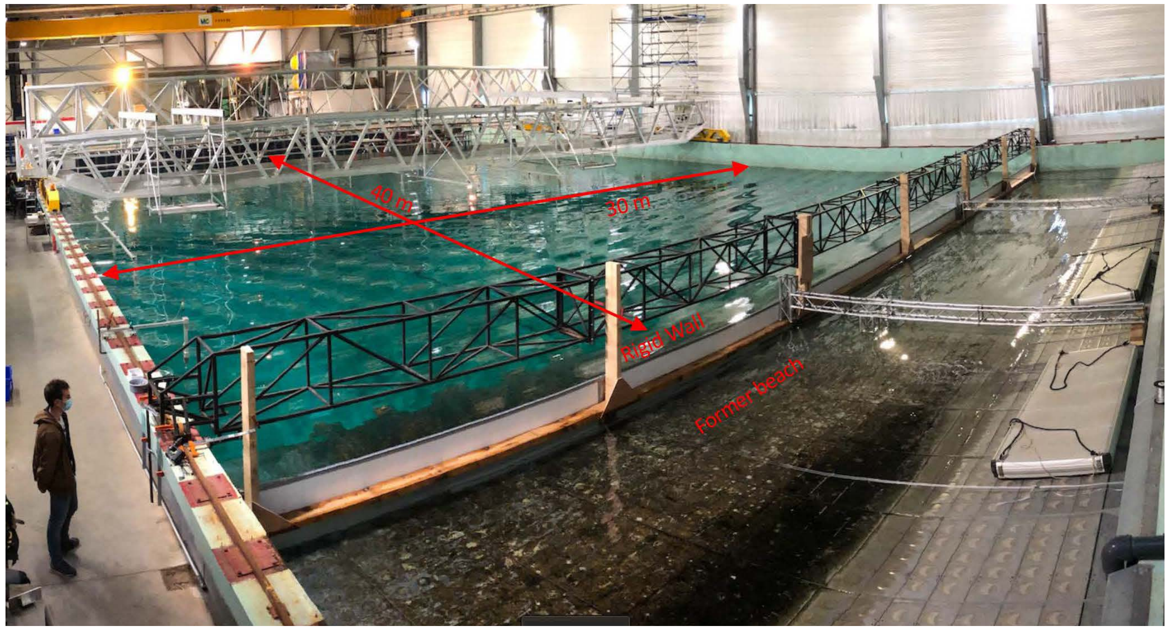
Nicolas Mordant, Eric Falcon Experiments in Surface Gravity–Capillary Wave Turbulence
Annual Review of Fluid Mechanics Vol. 54:1-25

Wave Turbulence experiments

Internal waves in stratified and/or rotating fluids

13m diameter, 100 tons of water,
2 tons of salt, rotation up to 2 rpm

Gravity-wave turbulence

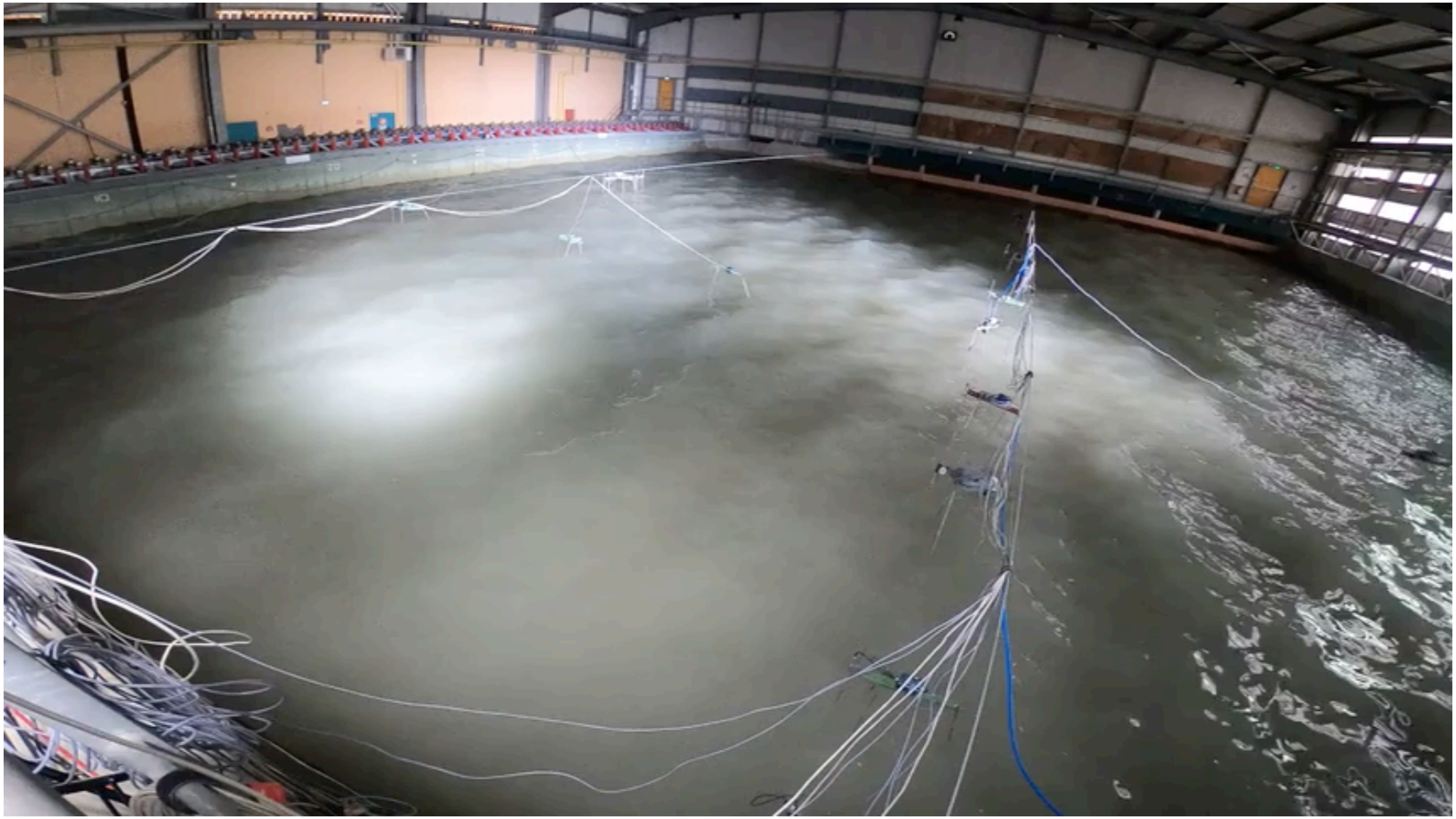


LHEEA CENTRALE NANTES
cnrs
université PARIS-SACLAY
Université de Paris
ENS
UGA Université Grenoble Alpes

Internal/inertial waves:
Grenoble (N. Mordant), Lyon (T. Dauxois), Orsay (P.P. Cortet), Paris (E. Falcon)

Wave Turbulence experiments

Experiment to study wave turbulence in shallow water



Random forcing JONSWAP (Joint North Sea Wave Project) spectrum at a peak "high" frequency (1 Hz) at the crossover deep/shallow water (weak turbulence)

Nicolas Mordant, Eric Falcon

Wave Turbulence experiments

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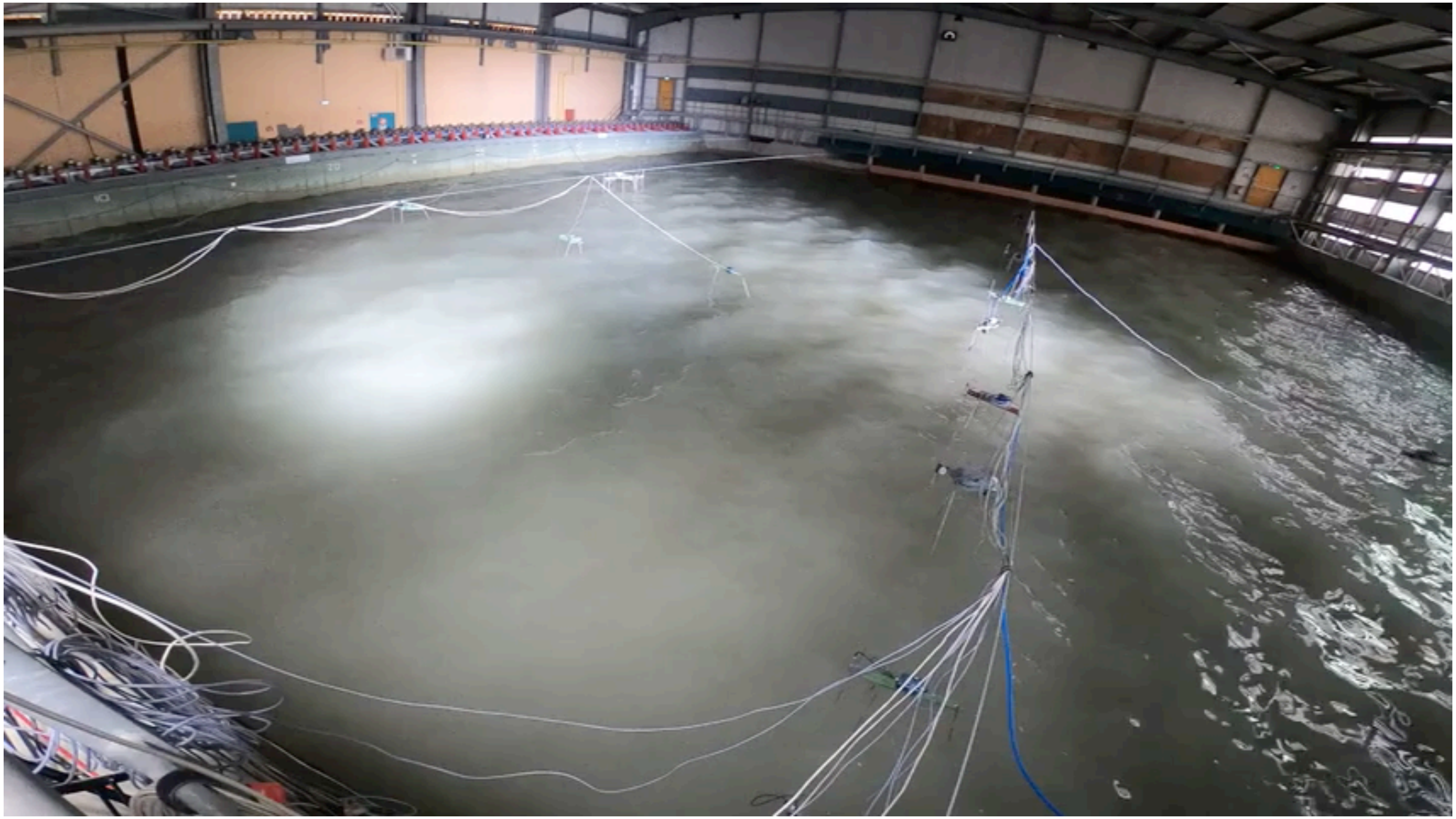

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Experiment to study wave turbulence in shallow water

Random forcing but at a peak frequency of forcing 0.2 Hz (shallow water regime) and a mix of dispersive waves and solitons

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Experiment to study wave turbulence in shallow water

Random forcing but at a peak frequency of forcing 0.2 Hz (shallow water regime) and a mix of dispersive waves and solitons

Heuristic derivation of WKE

Start with homogenous waves on $\mathbb{T}_L^d$

$$u(x,t) = \frac{1}{L^{\frac{d}{2}}} \sum_{k \in \mathbb{Z}_L^d} u_k(t) e^{ik\cdot x} \qquad u_k(0) = \phi_k g_k \qquad g_k \text{ iid Gaussian}$$

$$\partial_t u_k = -i\omega_k u_k + \frac{i\mu}{L^d} \sum_{S_k=0} M_{3k}^{12} u_{k_1} u_{k_2} \bar{u}_{k_3} \qquad S_k = k_1 + k_2 - k_3 - k$$

Factor the linear flow $u_k = e^{-i\omega_k t} a_k$

$$\partial_t a_k = \frac{i\mu}{L^d} \sum_{k_1,k_2,k_3} \delta(S_k) M_{3k}^{12} a_{k_1} a_{k_2} \bar{a}_{k_3} e^{-i\Omega t}$$

$$\Omega = \omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_k$$

Heuristic derivation of WKE

Expected value of the wave action $n_k \stackrel{def}{=} \mathbb{E}\left(|a_k|^2\right)$

$$\partial_t n_k = \frac{i\mu}{L^d} \sum_{k_1,k_2,k_3} \delta(S_k) M_{3k}^{12} \mathbb{E}\left(a_{k_1} a_{k_2} \bar{a}_{k_3} \bar{a}_k\right) e^{-i\Omega t} + C.C. \stackrel{def}{=} \frac{\mu}{L^d} I_4$$

$$\partial_t I_4 = \frac{\mu}{L^d} I_6$$

$$\partial_t I_6 = \frac{\mu}{L^d} I_8$$

$$\vdots$$

$$\partial_t I_{2j} = \frac{\mu}{L^d} I_{2j+2}$$

$$I_{2j}(0) = 0$$

Heuristic derivation of WKE

Solve on a time interval $[0, \Delta t]$

$$n_k(\Delta t) - n_k(0) = \frac{\mu^2}{L^{2d}} \sum \delta(S_k) \mathfrak{T} \frac{\sin^2(\Omega_k \Delta t/2)}{(\Omega_k/2)^2} + \mathcal{O}(\mu^4 \Delta t),$$

$$\mathfrak{T} = \left| M_{3,k}^{1,2} \right|^2 \left(\sum_{i=0}^{1} \frac{1}{n_{k_i}} - \sum_{i=2}^{3} \frac{1}{n_{k_i}} \right) \prod_{i=0}^{3} n_{k_i}$$

$L \to \infty$ converts the Riemann sum into an integral,

$$n_k(\Delta t) - n_k(0) = c_d \mu^2 \int \delta(S_k) \mathfrak{T} \frac{\sin^2(\Omega_k \Delta t/2)}{(\Omega_k/2)^2} dk_1 dk_2 dk_3 + \mathcal{O}(\mu^4 \Delta t)$$

HEURISTIC DERIVATION OF WKE— NLS $M^{1,2}_{3,k} = 1$

$$\Delta t \gg 1 \implies \frac{\sin^2(\Omega_k \Delta t/2)}{(\Omega_k/2)^2} \to 2\pi \Delta t \delta(\Omega_k)$$

$$n_k(\Delta t) - n_k(0) = \frac{c_d}{2\pi}\mu^2 \Delta t \int \delta(S_k)\delta(\Omega_k)\mathfrak{T} dk_1 dk_2 dk_3 + \mathcal{O}(\mu^4 \Delta t)$$

Assume $\mu^2 \Delta t \ll 1$ and defining $T_{kin} = \dfrac{2\pi}{c_d \mu^2}$, we get

$$\partial_\tau n_k = \int \delta(S_k)\delta(\Omega_k)\mathfrak{T} dk_1 dk_2 dk_3, \qquad \tau = \frac{t}{T_{kin}}$$

WKE (4-WAVES SYSTEM)

$$\partial_t n_k = \int \delta(S_k)\delta(\Omega_k)\left|M_{3,k}^{1,2}\right|^2 \left(\sum_{i=0}^{1} \frac{1}{n_{k_i}} - \sum_{i=2}^{3} \frac{1}{n_{k_i}}\right) \prod_{i=0}^{3} n_{k_i} dk_1 dk_2 dk_3$$
$$= \mathrm{Col}(n, k)$$

Energy conservation $$\frac{d}{dt}\int n_k \omega_k dk = 0$$

Wave action conservation $$\frac{d}{dt}\int n_k dk = 0$$

Entropy $$\frac{d}{dt}\int \ln(n_k) dk \geqslant 0 \qquad = 0 \iff \sum_{i=0}^{1} \frac{1}{n_{k_i}} = \sum_{i=2}^{3} \frac{1}{n_{k_i}}$$

WKE (4-WAVES SYSTEM)

Rayleigh-Jeans spectra $$n_k = \frac{T}{\mu + \omega_k}$$

Thermodynamic equilibrium $$\sum_{i=0}^{1} \frac{1}{n_{k_i}} = \sum_{i=2}^{3} \frac{1}{n_{k_i}}$$ zero flux

Are there equilibrium solutions with non zero flux? Simplify the problem by looking for Isotropic solutions

NLS in 3-d: $f(p,t) = 4\pi p^2 n_k(t), \quad p = |k|$

$$\frac{\partial f(p,t)}{\partial t} = 2\pi \int \min(p, p_1, p_2, p_3)(p\, p_1 p_2 p_3)^{-1} \delta(p_1^2 + p_2^2 - p_3^2 - p^2)$$
$$\times f f_1 f_2 f_3 \left(\frac{p^2}{f} + \frac{p_1^2}{f_1} - \frac{p_2^2}{f_2} - \frac{p_3^2}{f_3} \right) dp_1 dp_2 dp_3$$

Kolmogorov Zakharov Spectra for the NLS

Isotropic WKE $\partial_t f_p = 4\pi p^2 \mathrm{Col}(f, p)$

Energy conservation $\partial_t (\omega f_p) = -\partial_p Q(f, p)$ Q *energy flux*

Wave action conservation $\partial_t f_p = -\partial_p P(f, p)$ P *wave action flux*

Are there stationary solutions where either P or Q are not zero?

Kolmogorov Zakharov Spectra for the NLS

Ansatz $\quad n_k = \dfrac{C}{\omega^{\alpha}} = \dfrac{C}{|k|^{2\alpha}}$

$$\mathrm{Col}(n,k) = 4\pi^3 C^3 k^{4-6\alpha} I(\alpha)$$

$$I(\alpha) = \int \left[\min\left(1, q_1, q_2, q_3\right)\right]^{1/2} \left(q_1^{\alpha} + q_2^{\alpha} - q_3^{\alpha} - 1\right) \left(q_1 q_2 q_3\right)^{-\alpha} \delta\left(q_1 + q_2 - q_3 - 1\right) dq_1 dq_2 dq_3$$

Evaluate $I(\alpha)$ using Zakharov's (singular) transformation.

$$I(\alpha) \to I_{ZT}(\alpha), \qquad I(\alpha) = I_{ZT}(\alpha)?$$

Kolmogorov Zakharov Spectra for the NLS

Zakharov's transformation gives two possibilities

$$\alpha = 7/6 \qquad \text{and} \qquad \alpha = 3/2$$

$\alpha = 7/6$ $\quad Q(k) = 8\pi^4 C^3 |k|^{7-6\alpha} \frac{I_{ZT}(\alpha)}{3\alpha - 7/2}$ $\quad$ *Inverse cascade*

$\alpha = 3/2$ $\quad P(3/2) = \infty$ $\quad$ *Direct cascade*

Ying Zhu, Boris Semisalov, Giorgio Krstulovic, and Sergey Nazarenko
Phys. Rev. Lett. **130**, 133001 – Published 27 March 2023

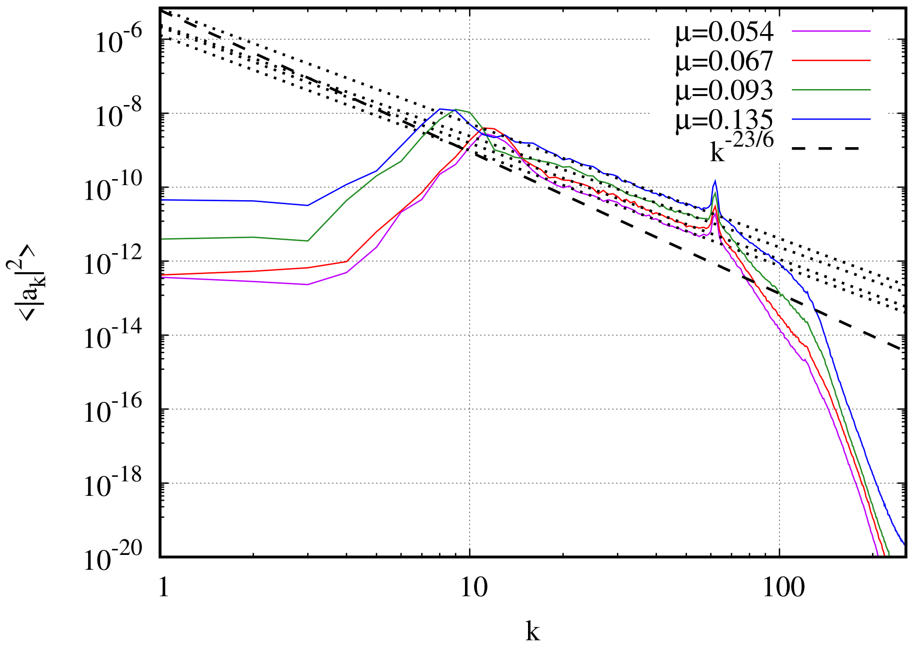
WTT and Gravity waves

$$\frac{\partial n_k}{\partial t} = \mathrm{Col}(n,k) + F_{pump}(k) - F_{dis}(k)$$

Gravity waves system is characterized by a dual cascade behavior

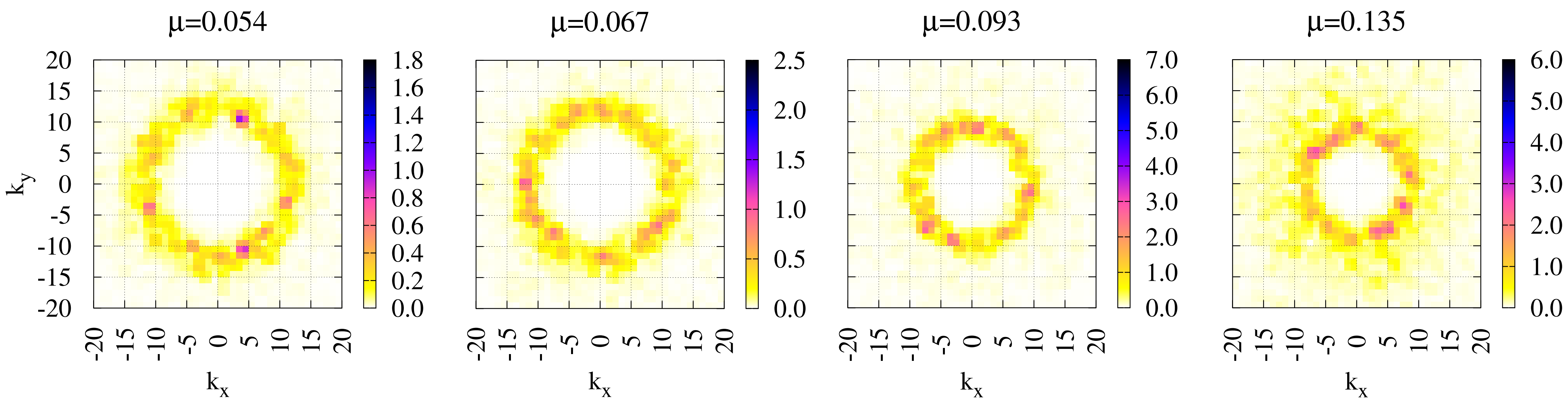
KZ spectra $n_k^{DC} \sim \frac{1}{|k|^4}$ energy cascade downscale

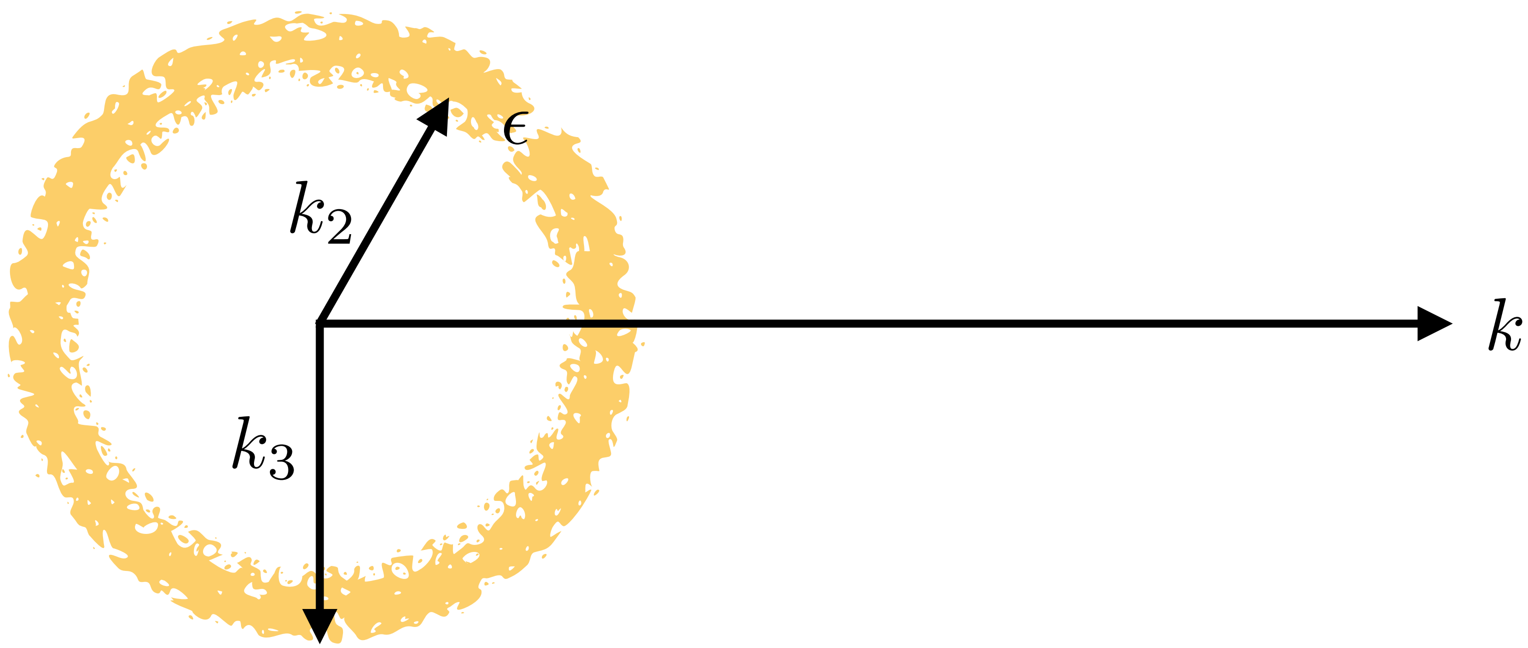
Second KZ spectra $n_k^{IC} \sim \frac{1}{|k|^{23/6}}$ the wave action is cascading upscale



Alexander O. Korotkevich PRL
Inverse cascade spectrum of gravity waves in the presence of a condensate: A direct numerical simulation

WTT and Gravity waves





Calculated constant flux solutions $\sim \dfrac{1}{|k|^3}$

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Inverse cascade spectrum of gravity waves in the presence of a condensate: A direct numerical simulation

WTT and Gravity waves

$$\frac{\partial n_k}{\partial t} \approx 8\pi \int \left\{ \left| T_{k,k_3}^{k_1,k_2} \right|^2 n_{k_2} n_{k_3} (n_{k_1} - n_k) \delta(\Omega) \right\}_{k_1 = k+q} dk_2 dk_3$$

$$\frac{\partial n_k}{\partial t} = 4\pi \int \delta(\Omega) n_{k_1} n_{k_3} \nabla_k \cdot \left\{ (k_2 - k_3) \left| T_{k,k_1}^{k,k_3} \right|^2 (k_2 - k_3) \cdot \nabla_k n_k \right\} dk_2 dk_3$$

Isotropic case $\dfrac{\partial}{\partial t}(2\pi p n_p) = \dfrac{\partial}{\partial p}\left(D_p \dfrac{\partial n_p}{\partial p} \right), \quad p = |k|, \quad D_p \sim p^4$

Isotropic wave action flux $n(k) = \dfrac{C}{|k|^3}$

A. Korotkevich, S. Nazarenko, Y. Pan, and J. Shatah:
Nonlocal gravity wave turbulence in presence of condensate.

RIGOROUS DERIVATION OF WKE

$$\partial_t a_k = \frac{i\mu}{L^d} \sum_{k_1,k_2,k_3} e^{-i\Omega t} a_{k_1} \bar{a}_{k_2} a_{k_3} \delta(S_k)$$

The waves k_1, k_2, k_3, k are resonant if

$$\Omega = \omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_k = 0$$

$$\partial_t a_k = \frac{i\mu}{L^d} \sum_{\Omega=0} a_{k_1} \bar{a}_{k_2} a_{k_3} \delta(S_k) + \frac{i\mu}{L^d} \sum_{\Omega\neq 0} e^{-i\Omega t} a_{k_1} \bar{a}_{k_2} a_{k_3} \delta(S_k)$$

Iterate Duhamel's formula: first term grows in time. The second term oscillates

Range of parameters

1) Weakly nonlinear $\mu \ll 1$. 2) Large domain $L \gg 1$.

3) Long time intervals $\Delta t \gg 1$

The range of parameters where WKE gives a good description.

1) Linear time scale $T_{lin} = \frac{1}{\omega_k} = \mathcal{O}(1)$

2) Nonlinear time scale $T_{nl} = \mathcal{O}(\frac{1}{\mu})$.

3) Resonant time scale $T_{res} = \mathcal{O}(L^2)$

4) Kinetic time scale $T_{kin} = \mathcal{O}(\frac{1}{\mu^2})$

$$T_{lin} \ll T_{nl} \leqslant T_{kin} \ll T_{res}$$

$$L \to \infty : \quad \frac{1}{L} \leqslant \mu \ll 1, \quad \tau = \frac{t}{\mu^2}, \quad \mu = \frac{1}{L^\alpha}, \quad 0 < \alpha < 1,$$

Derivation of WKE

1. *Developing a series by iterating Duhamel's formula (integrate by parts in time) requires keeping track of how many terms and the size of each term (as a function of L).*
   Can be done by very difficult combinatorial, probabilistic and number theoretic arguments.

2. *Converting the sum into on integral: Requires Fourier analysis and number theory.*
   To get the full range of parameters when this is possible requires deep results in number theory.

3. *Controlling the remainder.*
   This require hard PDE analysis and optimal estimates on the iterates.

This program was carried out by ***Y. Deng and Z. Hani*** *for the NLS.*

Rigorous Results

Periodic setting

Buckmaster–Germain–Hani–Shatah
Deng–Hani, Collot–Germain
Deng–Hani
Staffilani–Tran (stochastic forcing), Ma (dissipation)

Related results

Erdös–Salmhofer–Yau *(Linear Schrödinger equation)*
Lukkarinen–Spohn *(Gibbs measure)*

Deng–Hani (*NLS on the periodic box* $\mathbb{T}^d_L$ *with* $d > 3$)

There exists $\delta < 1$ *fixed, such that for L large enough*

$$\mathbb{E}|\widehat{u}_k(t)|^2 \to n(\frac{t}{T_{kin}}, k)$$

uniformly in (t, k) for

$$t \in [0,\ \delta T_{kin}] \qquad \mu = \frac{1}{L^\alpha}, 0 < \alpha \leqslant 1$$

The iterates: Feynman's trees

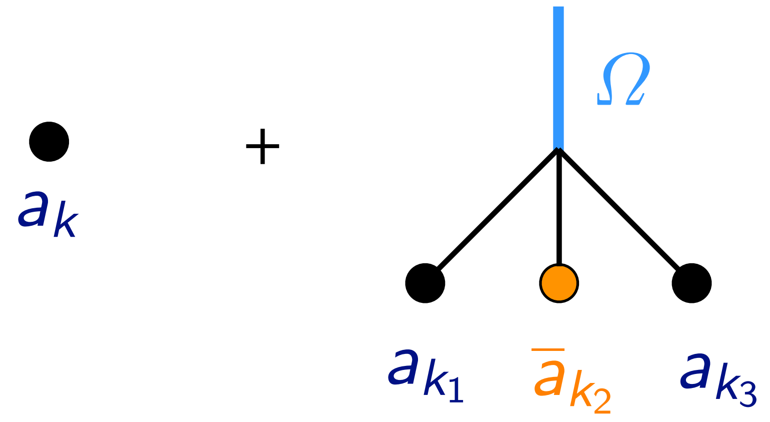
$$\partial_t a_k = \frac{i\mu}{L^d} \sum_{k_1,k_2,k_3} e^{-i\Omega t} a_{k_1} \overline{a}_{k_2} a_{k_3} \delta(S_k)$$

Laplace transform

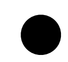
$$s\mathscr{L}(a_k)(s) = a_k(0) + \frac{i\mu}{L^d} \sum_{k_1,k_2,k_3} \int_0^\infty e^{-(s+i\Omega)t} a_{k_1} \overline{a}_{k_2} a_{k_3} \delta(S_k)$$

Integrate by parts

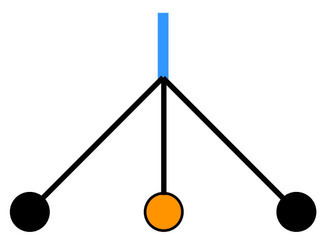
$$a_k + \frac{1}{s+i\Omega} a_{k_1} \overline{a}_{k_2} a_{k_3} \delta(S_k) + \ldots$$



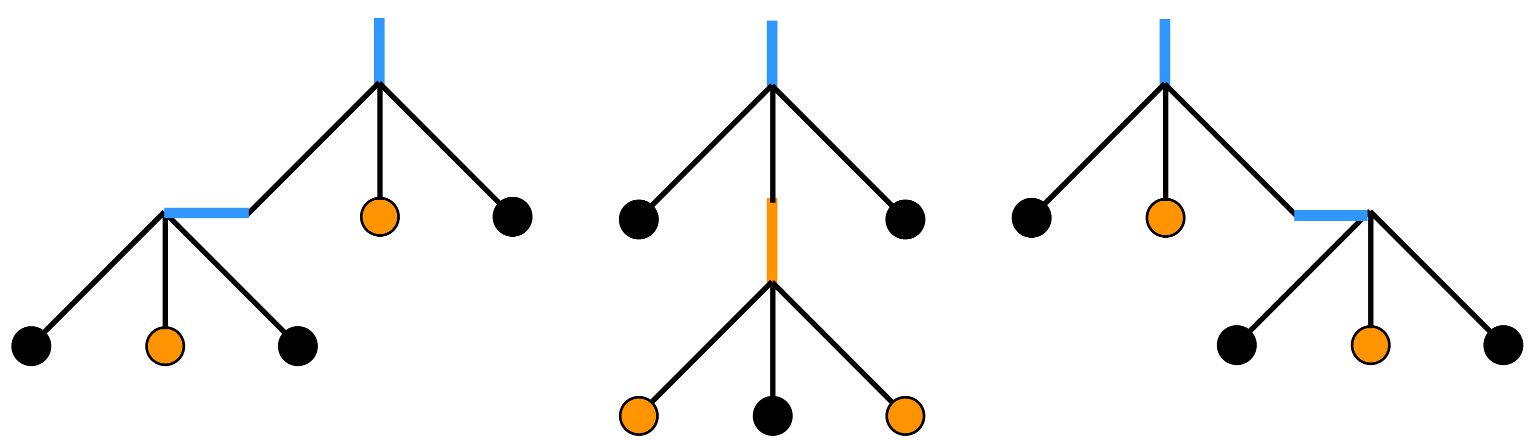
Tree representation



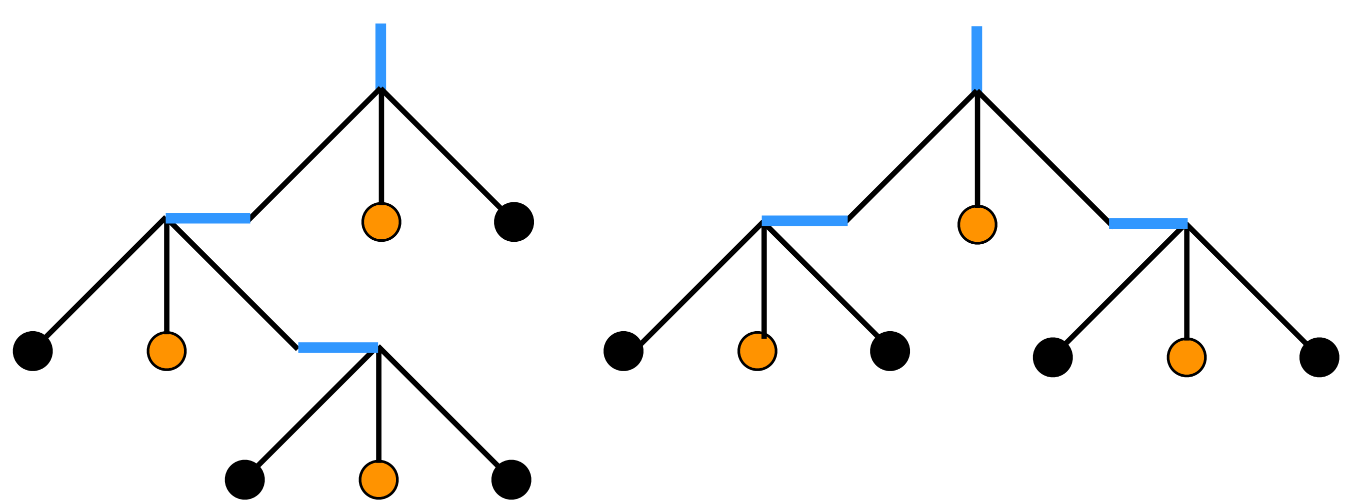
$$+ \quad \left(\frac{i\mu}{L^d}\right)$$



$$+ \quad \left(\frac{i\mu}{L^d}\right)^2$$



$$+ \quad \left(\frac{i\mu}{L^d}\right)^3$$



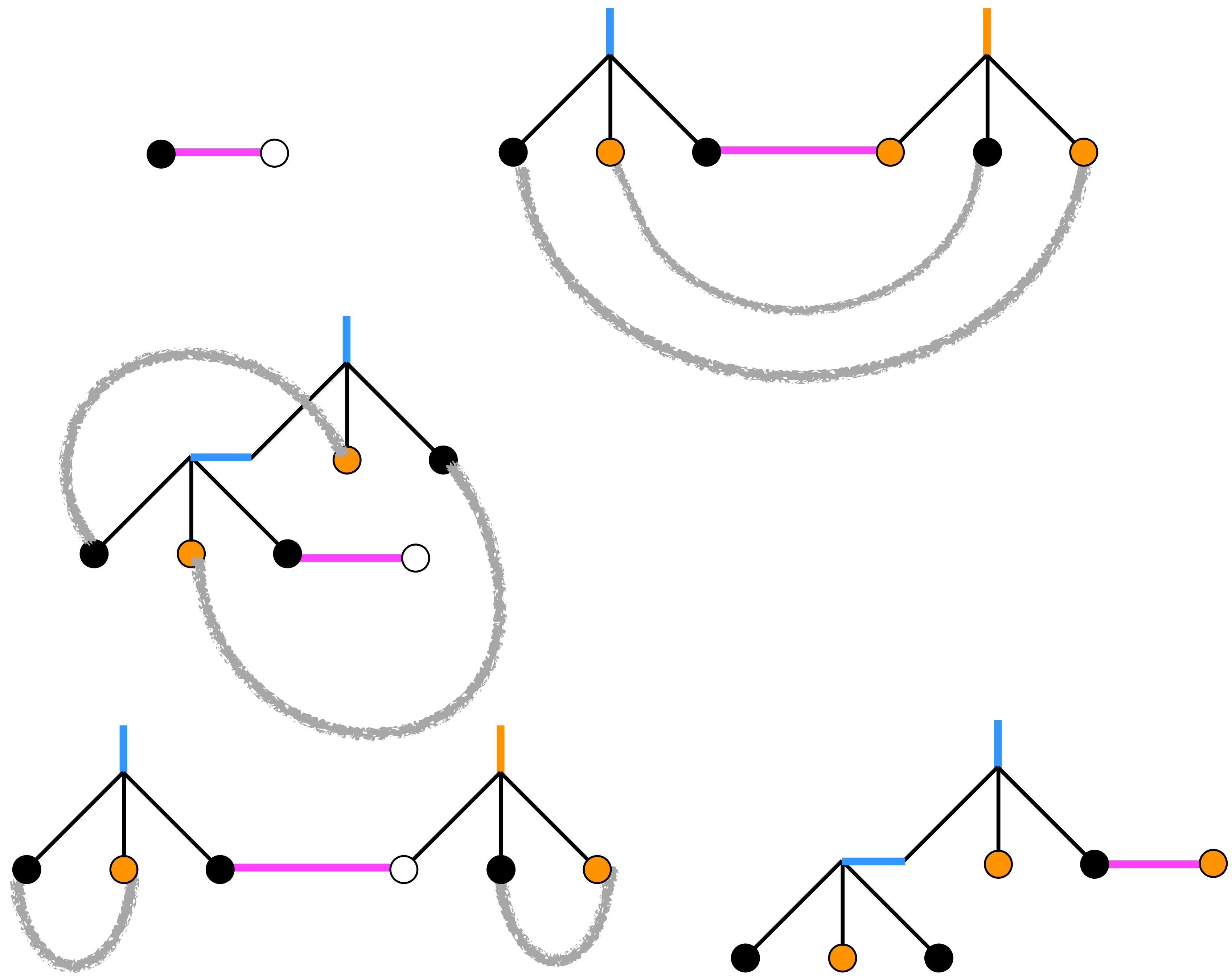
$+ \quad \cdots$

Tree pairings: regular vs self-pairing vs degenerate

Regular

Self interacting

Degenerate



Tree pairings

(Regular)

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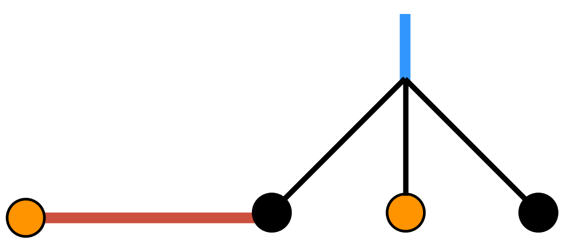

(Degenerate)

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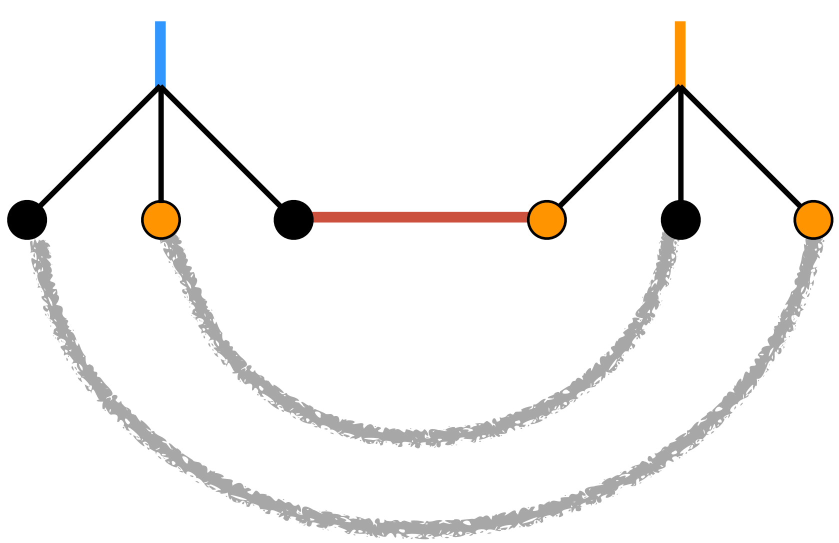

(Regular)

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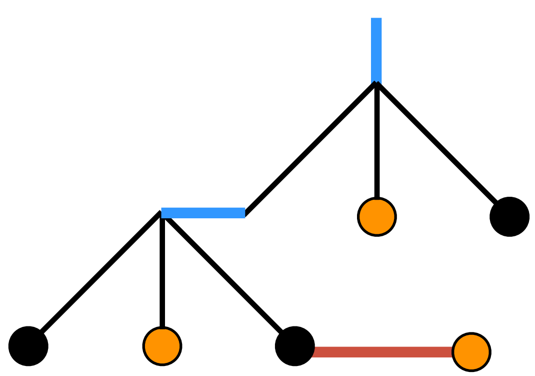

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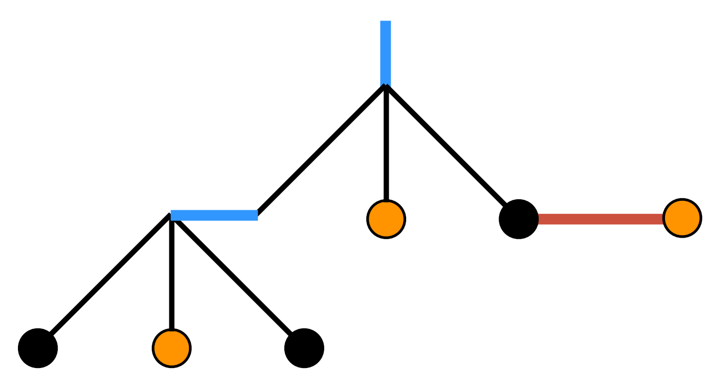
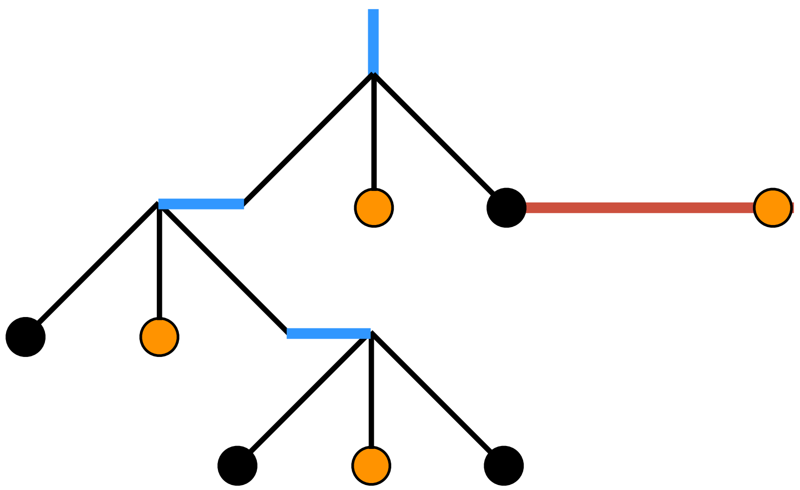


(Self-interacting or degenerate)

Tree pairings



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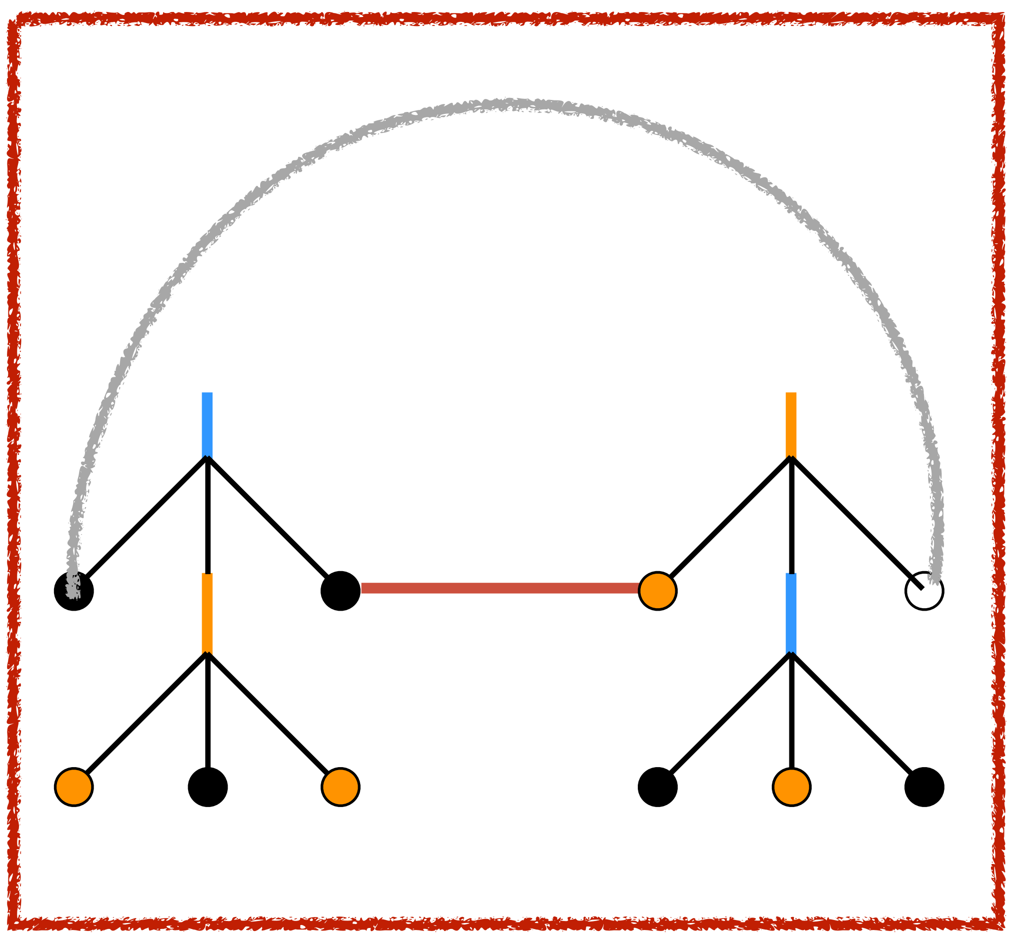

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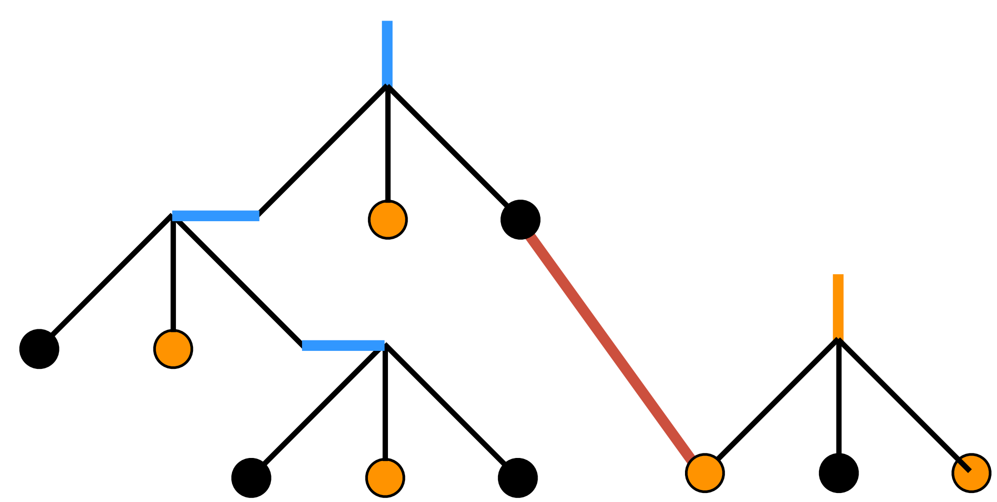

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REMARKS

1. Wave turbulence regimes are proven to exist naturally, experimentally, and mathematically.
    a. Gravity wave turbulence
    b. Capillary wave turbulence

2. Range of applicability is work in progress

3. Extensions: Many situations are still unclear (interaction with finite- amplitude waves)

4. Fertile area to develop new analytical techniques for solving PDEs

Inhomogeneous turbulence in $\mathbb{R}^d$

Inhomogeneous waves,

$$u(x,t) = \frac{1}{L^{\frac{d}{2}}} \sum_{k \in \mathbb{Z}_L^d} u_k(\epsilon x, t) e^{ik\cdot x}$$

$\epsilon > 0$ is the scale of the spectral width of wave packets.

Random data: $u_0(x) = \frac{1}{L^{\frac{d}{2}}} \sum \phi_k(\epsilon x) g_k e^{ik\cdot x}$, g_k *i.i.d. Gaussians*

Wave action density

$$W(x,\zeta,t) = \boldsymbol{\mathcal{W}}(u) = \int e^{-i2\pi\zeta\cdot y} \mathbb{E}(u(x+\frac{y}{2},t)\bar{u}(x-\frac{y}{2},t))dy$$
$$= \int e^{i2\pi x\cdot\eta} \mathbb{E}(\widehat{u}(\zeta+\frac{\eta}{2},t)\bar{\widehat{u}}(\zeta-\frac{\eta}{2},t))d\eta$$

Derive WKE for $W(x,\xi,t)$

Homogenous vs non homogenous

Homogenous

Pairing rule: $A_k(t)$ is paired with $\overline{A}_k(t)$

Self interacting trees can be bounded
And contribute to the WKE

Cancelation of irregular chains where each term considered separately can be unbounded

Inhomogenous

Pairing rule: $\hat{A}_k(\zeta + \eta/2, t)$ is paired with $\overline{\hat{A}}_k(\zeta - \eta/2, t)$

Self interacting trees Can not be bounded. We have divergence of self-interacting trees.

No cancelation of irregular chains

Related work: Ampatzoglou–Collot–Germain (quadratic nonlinearity)
Hannani–Rosenzweig–Staffilani–Tran (stochastic forcing)

Model equation

Would like to have a model where

1. We can rigorously formulate and test the theoretical foundations of inhomogeneous WTT.
2. Simple tree pairing structure that eliminates self-interactions.
3. Exactly solvable using perturbation methods.

Wick nonlinear Schrödinger equation (WNLS) *H. Zhu, Z. Hani, and J. S.*

$$i\partial_t u + \Delta u + \lambda u \odot \overline{u} \odot u = 0$$

WNLS

$\{g_k\}$ are Gaussian random variables with zero mean.

$\mathscr{A}$ probability space: σ-algebra generated by $\{g_k\}$

$$P_p = \overline{\{\text{Set of polynomials in } \{g_k\} \text{ of degree} \leqslant p.\}} \quad \text{(Closure in } L^2(\mathscr{A})\text{)}$$

$$H^{:p:} = P_{p-1}^{\perp} \cap P_p$$

$$L^2(\mathscr{A}) = \bigoplus_{p \geqslant 0} H^{:p:}$$

π_p : orthogonal projection on $H^{:p:}$

$$X \odot Y = \pi_{p+q}(XY), \quad \text{if } X \in H^{:p:},\ Y \in H^{:q:}$$

WNLS

$$i\partial_t u + \Delta u + \lambda u \odot \overline{u} \odot u = 0$$

$$u = \sum_{p=0}^{\infty} u^p, \quad u^p = \pi_p u$$

$$i\partial_t u^p + \frac{1}{4\pi}\Delta u^p = -\lambda \sum_{p_1+p_2+p_3=p} u^{p_1} \odot \overline{u^{p_2}} \odot u^{p_3}, \quad p \in \mathbb{N}_0.$$

Well prepared data: narrow wave packets and phase randomization

$$u(0,x) = \frac{1}{L^{d/2}} \sum_{k\in\mathbb{Z}_L^d} \phi(\epsilon x, k) e^{2\pi i k\cdot x} g_k$$

CORRELATIONS

Wave kinetic equation $\quad L \to \infty, \quad \mu^2 = \frac{1}{L^a}, \quad \epsilon = \frac{1}{L^b}, \quad a = b$

$$(\partial_t + 2k \cdot \nabla_x) W_k = 2 \int_{\mathscr{D}_k} \delta(\Omega_k) \prod_{1 \leqslant j \leqslant 3} W_{k_j} dk_1 dk_2 dk_3, \quad k \in \mathbb{R}^d, \quad a = b$$

$$\mathbb{E}\left|\Pi_p a_k\right|^2 (t/\mu^2, y) \to W_k(t, y), \quad \Pi_p = \sum_{q=0}^{p} \pi_q$$

$$\lim_{p \to \infty} \lim_{L \to \infty} \sup_{|t| \leqslant T} \sup_{k \in \mathbb{Z}_L^d} \left| \mathbb{E}\left|\Pi_p a_k\right|^2 (t/\mu^2, \cdot) - W_k(t, \cdot) \right|_{L^\infty} \to 0$$

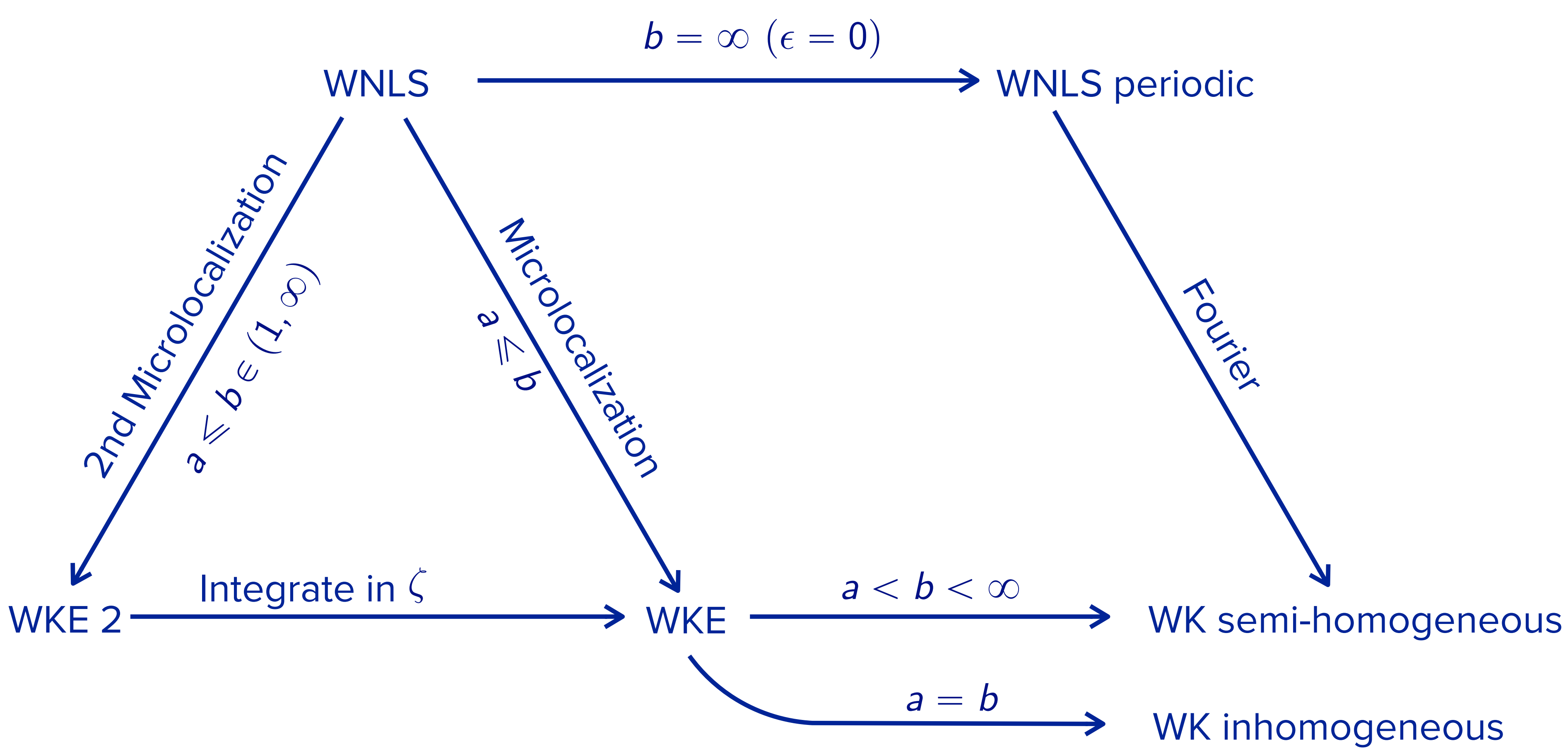
H. Zhu, Z. Hani, and J. Shatah

SECOND MICRO-LOCALIZATION

Wigner transform of $\Pi_p A_k$

$$\mathbb{E}\mathcal{W}\left[\Pi_p A_k\left(t/\mu^2\right)\right](y,\zeta) \to E_{k,\zeta}(t,y)$$

$$(\partial_t + \nabla_k \omega \cdot \nabla_x) E_{k,\zeta} = 2 \int_{\mathscr{D}_k} \delta(\Omega(k)) \left(\int_{\mathscr{D}_\zeta} \prod_{1 \leqslant j \leqslant 3} E_{k_j,\zeta_j} d\zeta \right) dk, \quad (k,\zeta) \in \mathbb{R}^{2d},$$



Thank you for your attention.