Generic alignment conjecture and the problem of emergence of collective behavior

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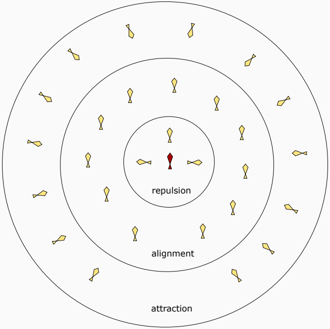
Abel Symposium 2023

Reynold 3Zone model



1. Collision Avoidance:
avoid collisions with nearby flockmates

2. Velocity Matching: attempt
to match velocity with nearby flockmates

3. Flock Centering:
attempt to stay close to nearby flockmates

Published in *Computer Graphics*, **21**(4), July 1987, pp. 25-34.
(ACM SIGGRAPH '87 Conference Proceedings, Anaheim, California, July 1987.)

**Flocks, Herds, and Schools:
A Distributed Behavioral Model**

Craig W. Reynolds

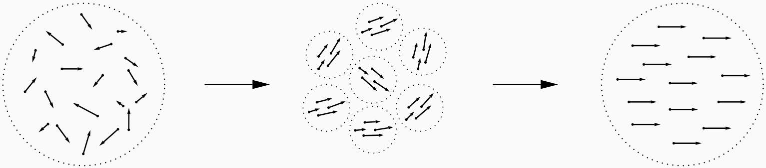
Symbolics Graphics Division

Discrete models of collective behavior describe dynamics of a number of agents:

$$\dot{\mathbf{x}}_i = \mathbf{v}_i,$$
$$\dot{\mathbf{v}}_i = \mathrm{s}_i([\mathbf{v}]_i - \mathbf{v}_i), \qquad (\mathbf{x}_i, \mathbf{v}_i) \in \Omega \times \mathbb{R}^n.$$

Agents adjust their directions to environmentally averaged velocity $[\mathbf{v}]_i$ with a strength s_i.

Emergence is formation of global patterns resulting from local interactions.



Cucker-Smale model

- Cucker-Smale alignment model, 2007:

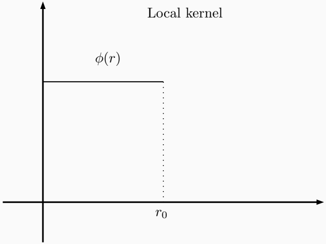
$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \phi(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{v}_j - \mathbf{v}_i), \end{cases} \qquad (\mathbf{x}_i, \mathbf{v}_i) \in \Omega \times \mathbb{R}^n$$

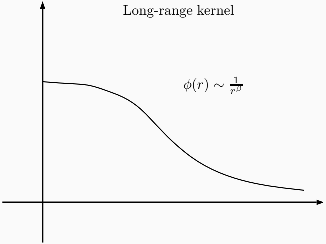
where

$$\mathrm{s}_i = \frac{1}{N}\sum_j \phi(\mathbf{x}_i - \mathbf{x}_j), \quad [\mathbf{v}]_i = \frac{\sum_j \phi(\mathbf{x}_i - \mathbf{x}_j)\mathbf{v}_j}{\sum_j \phi(\mathbf{x}_i - \mathbf{x}_j)}$$

and ϕ is a radially decreasing communication kernel.

- Sufficiently strong global communication $\Rightarrow$ Unconditional alignment.





Theorem (Cucker, Smale (2007); Ha, Tadmor (2008); Ha, Liu (2009))

Suppose

$$\phi(r) = \frac{H}{(1+r^2)^{\frac{\beta}{2}}}.$$

if $\beta \leqslant 1$, then any solution aligns and flocks exponentially fast:

$$\max_i |\mathbf{v}_i - \bar{\mathbf{v}}| \leqslant Ce^{-\delta t}, \quad \sup_{i,j} |\mathbf{x}_i - \mathbf{x}_j| \leqslant \bar{D} < \infty.$$

– kinetic version due to Carrillo, Fornasier, Rosado, Toscani (2010);

– macroscopic version due to Tan, Tadmor (2014).

CS in applications

– L. Perea, P. Elosegui, and G. Gomez. Extension of the Cucker-Smale control law to space flight formations. Journal of Guidance, Control, and Dynamics, 32:526 – 536, 2009.

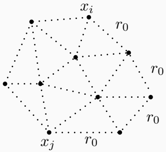
– M.Bongini, M.Fornasier and D.Kalise, (Un)conditional consensus emergence under perturbed and decentralized feedback controls, Discr. Contin. Dyn. Syst. Ser. A 35 (2015) 4071–4094.

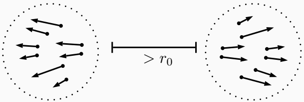
– Y.-P. Choi, D. Kalise, J. Peszek and A. A. Peters, A collisionless singular Cucker–Smale model with decentralized formation control, SIAM J. Appl. Dyn. Syst. 18 (2019), no. 4, 1954–1981.

– J.-A. Carrillo, Y.-P. Choi, C. Totzeck and O. Tse, An analytical framework for consensus-based global optimization method, Math. Models Methods Appl. Sci. 28 (2018) 1037–1066.

– Zhiping Mao, Zhen Li, George Em Karniadakis, Nonlocal flocking dynamics: Learning the fractional order of PDEs from particle simulations. Commun. Appl. Math. Comput. 1 (2019), no. 4, 597–619.

"Local communication ⇒ Flocking" commonly requires propagation of connectivity.





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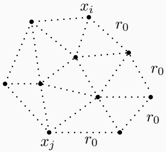

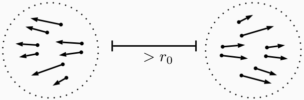

Theorem (Morales, Peszek, Tadmor (2019))

If the flock remains r_0-connected at all times, then it aligns. If the kernel is local, $\phi(r) \sim \Lambda \mathbb{1}_{r<r_0}$ but strong, $\Lambda \gg 1$, then any initially r_0-connected data results in aligned outcome.

Theorem (Tadmor, Shu (2019))

Under quadratic confinement force $F = -U(x)$, $U(r) = r^2$, the condition on ϕ relaxes to $\phi \gtrsim \frac{1}{r^2}$.

Let us assume purely local communication:

$$\phi(x) \geqslant \lambda \mathbb{1}_{|x| \leqslant r_0}.$$

Attempted approaches:

– microscopic level: generic alignment conjecture.

– kinetic level: alignment = problem of relaxation for kinetic models; hypocoercivity method.

– macroscopic level: hydrodynamic connectivity; how low can the density ρ be to ensure collective outcome?

Let us assume only local communication: "locked states"

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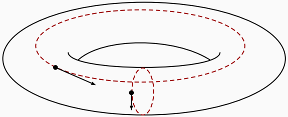

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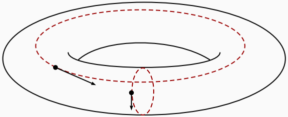

Kronecker, 1800's: The Euclidean line $\mathbf{x} = t\mathbf{x}_0 + \mathbf{v}_0$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_N), \qquad \mathbf{v} = (v_1, v_2, \ldots, v_N)$$

densely fills a k-dimensional subtorus of $\mathbb{T}^{nN}$ where

$$k = \dim_{\mathbb{Q}} \sum_{j=1}^{nN} \mathbb{Q}\mathbf{v}_0^j.$$

Generic Alignment Conjecture on $\mathbb{T}^n$

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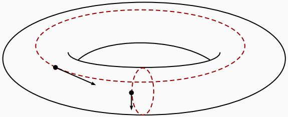

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Generic Alignment Conjecture: For almost every initial data $(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{T}^{nN} \times \mathbb{R}^{nN}$ solutions to the Cucker-Smale system align

$$\max_{i,j} |v_i - v_j| \to 0.$$

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$$\phi(x) = \begin{cases} \infty, & |x| \leqslant r_0; \\ 0, & |x| > r_0. \end{cases} \tag{1}$$

RS (2023) The conjecture is true for $N = 2$ and any $n \in \mathbb{N}$. In fact, for any N, for almost every initial data at least 2 agents will align. Moreover, the ensemble dynamics contracts volumes to 0:

$$\det \nabla S_t(\mathbf{x}, \mathbf{v}) = \exp \left\{ -\int_0^t \sum_{i \neq j} \phi(x_i - x_j)\, \mathrm{d}s \right\} \to 0.$$

Large crowd dynamics

S.-Y. Ha, E. Tadmor (2008); S.-Y. Ha , J.-G. Liu (2009): mean-field limit

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{v_i} \otimes \delta_{x_i} \to f$$

$$\partial_t f + v \cdot \nabla_x f = \mathrm{s}_\rho \nabla_v ((v - [u]_\rho) f), \qquad \text{(Vlasov-Alignment)}$$

where ρ and u are the macroscopic variables

$$\rho(x,t) = \int_{\mathbb{R}^n} f(x,v,t)\, \mathrm{d}v, \quad \rho u = \int_{\mathbb{R}^n} v f(x,v,t)\, \mathrm{d}v,$$

and s_ρ and $[u]_\rho$ come from the model in question:

$$\mathrm{s}_\rho = \rho_\phi = \rho * \phi, \qquad [u]_\rho(x) = \frac{(u\rho)_\phi}{\rho_\phi} \qquad \text{(Cucker-Smale)}$$

$$\mathrm{s}_\rho = 1, \qquad [u]_\rho(x) = \frac{(u\rho)_\phi}{\rho_\phi} \qquad \text{(Motsch-Tadmor)}$$

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RS (2022) arxiv: Environmental Averaging.

Euler Alignment system

The macroscopic variables satisfy the Euler-Alignment system

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla_x \cdot (\rho u \otimes u + \mathcal{R}) = \int_{\mathbb{R}^n} \rho(x)\rho(y)(u(y) - u(x))\phi(x-y)\,\mathrm{d}y, \end{cases}$$

where $\mathcal{R}$ is the Reynolds stress tensor,

$$\mathcal{R}(x,t) = \int_{\mathbb{R}^n} (v - u(x,t)) \otimes (v - u(x,t)) f(x,v,t)\,\mathrm{d}v.$$

Kang, Vasseur (2014); Figalli, Kang, (2019); RS (2020): hydrodynamic limit in monokinetic regime

$$f \to \rho(x,t)\delta(v - u(x,t))$$

This leads to pressureless EAS, $\mathcal{R} = 0$.

Karper, Mellet, Trivisa, 2014-16: Maxwellian regime

$$f \to \frac{\rho(x,t)}{(2\pi)^{n/2}} e^{-\frac{|v-u(x,t)|^2}{2}}$$

This leads to isothermal pressure, $\mathcal{R} = \rho\mathbb{I}$.

Well-posedness of hydrodynamic models

– Carrillo, Choi, Tadmor, Tan (2014-2016). Pressureless EAS: g.w.p. under critical threshold criterion, smooth kernel 1D:

$$e = u_x + \phi * \rho \geqslant 0, \quad e_t + (eu)_x = 0.$$

– Tadmor, RS, (2016-2017); T. Do, A. Kiselev, L. Ryzhik, and C. Tan (2017): singular fractional parabolic models, unconditional g.w.p. in 1D

$$\phi(r) = \frac{1}{r^{1+\alpha}}, \quad 0 < \alpha < 2.$$

(Caffarelli-Chen-Vasseur, Caffarelli-Silvestre regularization; alternatively, modulus of continuity)

– He, Tadmor (2016) spectral dynamics in 2D, D. Lear, RS (2019-21) unidirectional flows, Ch. Tan (2020) radial flows, Danchin, Mucha, Peszek, Wroblewski (2019), RS (2019) small initial data, etc.

– Choi (2018) 1D isothermal EAS, small data; Constantin, Drivas, RS 2020 global well-posedness of non-vacuous solutions, entropy hierarchy.

Hydrodynamic approach. Spectral Gap

Consider solution to pressureless EAS

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla_x \cdot (\rho u \otimes u) = \int_{\mathbb{R}^n} \rho(x)\rho(y)(u(y) - u(x))\phi(x-y)\,\mathrm{d}y, \end{cases}$$

with $\bar{u} = \int_\Omega \rho u \,\mathrm{d}x = 0$. Then alignment can be measured by the energy

$$\mathcal{E} = \frac{1}{2}\int_{\Omega\times\Omega} \rho|u|^2\,\mathrm{d}x.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = -\frac{1}{2}\int_{\Omega\times\Omega} |u(x,t) - u(y,t)|^2 \rho(x,t)\rho(y,t)\phi(x-y)\,\mathrm{d}x\,\mathrm{d}y := -(u, \mathcal{L}_\rho u)_\rho,$$

where $\mathcal{L}_\rho u = \mathrm{s}_\rho(u - [u]_\rho)$. Let

$$\lambda = \inf_{u\in L^2_0(\rho)} \frac{(u, \mathcal{L}_\rho u)_\rho}{(u,u)_\rho} \geqslant 0. \tag{2}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leqslant -\lambda\mathcal{E} \quad \Rightarrow \quad \mathcal{E} \to 0 \text{ if } \int_0^\infty \lambda(s)\,\mathrm{d}s = \infty.$$

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Tadmor (2021): $\lambda \sim \frac{\rho_-^2}{\rho_+}$. So, provided ρ_+ is under control we need $\rho_- \gtrsim \frac{1}{\sqrt{t}}$.

Low Energy Method

Let us rewrite the energy equality in a different way:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = (u, [u]_\rho)_{\rho \mathrm{s}_\rho} - (u, u)_{\rho \mathrm{s}_\rho} := \mathcal{E}_1 - \mathcal{E}_0.$$

Need:

$$\mathcal{E}_0 - \mathcal{E}_1 \geqslant \varepsilon \mathcal{E}_0 \gtrsim \varepsilon \rho_- \mathcal{E}.$$

So, then $\lambda \sim \varepsilon \rho_-$. Can we find a bound on ε independent of ρ_+?

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$$\begin{aligned}
\mathcal{E}_0 &= (u, u)_{\rho \mathrm{s}_\rho} \\
\mathcal{E}_1 &= (u, [u]_\rho)_{\rho \mathrm{s}_\rho} \\
\mathcal{E}_2 &= ([u]_\rho, [u]_\rho)_{\rho \mathrm{s}_\rho} \\
\mathcal{E}_3 &= ([u]_\rho, \left[[u]_\rho\right]_\rho)_{\rho \mathrm{s}_\rho} \ldots
\end{aligned}$$

It turns out:

$$\mathcal{E}_1 - \mathcal{E}_2 \geqslant \varepsilon \mathcal{E}_1 \quad \Rightarrow \quad \mathcal{E}_0 - \mathcal{E}_1 \geqslant \varepsilon \mathcal{E}_0.$$

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KEY observation: estimates on the lower energy gap are independent of ρ_+!

For Cucker-Smale: supposing $\phi = \psi * \psi$, we have where $\psi \geqslant 0$, alignment occurs provided

$$\mathcal{E}_1 - \mathcal{E}_2 = \frac{1}{2}\int_{\Omega^2} \rho_\psi(x)\rho_\psi(y)\left(\frac{\rho}{\rho_\phi}\right)_{\psi\psi}(x,y)\left|\frac{(u\rho)_\psi}{\rho_\psi}(x) - \frac{(u\rho)_\psi}{\rho_\psi}(y)\right|^2 \mathrm{d}y\,\mathrm{d}x,$$

where

$$r_{\psi\psi}(x,y) = \int_\Omega \psi(x-\xi)\psi(y-\xi)r(\xi)\,\mathrm{d}\xi,$$

This leads to the new estimate

$$\lambda \sim \rho_-^4.$$

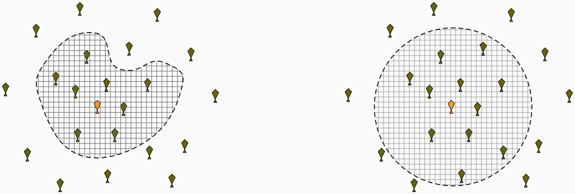
So, for alignment we need

$$\rho_- \gtrsim \frac{1}{\sqrt[4]{t}}.$$

Topological diffusion

When communication has a limited range the interaction may be "topological" rather than "metric":

Topological versus Metric protocol



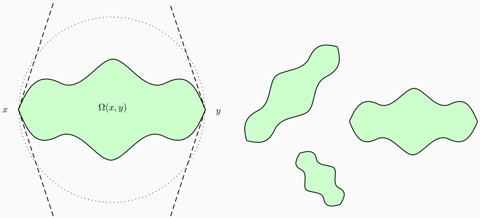
– A. Cavagna, A. Cimarelli, I. Giardina, A. Orlandi, G. Parisi, A. Procaccini, R. Santagati, E. Silvestri, F. Stefanini, and M. Viale, V. Zdravkovic (2008) : StarFlag Project.

– J. Haskovec. Flocking dynamics and mean-field limit in the Cucker-Smale type model with topological interactions. Phys. D, 261(15):42–51, 2013.

– A. Blanchet and P. Degond. Topological interactions in a Boltzmann-type framework. J. Stat. Phys., 163:41–60, 2016.

– A. Blanchet and P. Degond. Kinetic models for topological nearest-neighbor interactions. J. Stat. Phys. volume 169: 929–950, 2017.

– Tadmor, RS. Topologically based fractional diffusion and emergent dynamics with short range interactions. SIAM J. Math. Anal., 52(6):5792–5839, 2020



1. Every agent x has a finite influence range, $B(x, r_0)$.
2. Agent x influences agent y through communication domain $\Omega(x, y) = \Omega(y, x)$.
3. The mass
$$d(x, y, t) = \int_{\Omega(x,y)} \rho(z, t)\, \mathrm{d}z.$$
determines the communication distance between x and y.

Based on the outlined principles, we make the following choice:

$$\phi_\rho(x, y) = \frac{1}{d(x, y, t)|x-y|^\alpha} \mathbb{1}_{|x-y|<r_0}.$$

Topological Euler-Alignment system

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \displaystyle\int_{\mathbb{T}^n} \phi_\rho(x, y)(u(y, t) - u(x, t))\rho(y, t) \, \mathrm{d}y. \end{cases}$$

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Theorem (Tadmor, RS (2018))

Let (u,ρ) be a global smooth solution to the topological model on $\mathbb{T}^n$ and

$$\rho(x,t) \gtrsim \frac{1}{t}, \quad t \to \infty. \tag{3}$$

Then

$$|u(t) - \bar{u}|_\infty \lesssim \frac{1}{(\ln t)^{1/6}}.$$

In 1D the lower bound (3) holds automatically.

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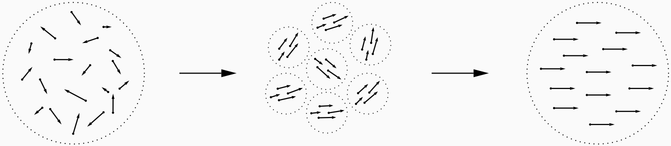

Well-posedness

RS, Tadmor. Topologically based fractional diffusion and emergent dynamics with short-range interactions, *SIMA*. Vol. 52, No. 6, pp. 5792–5839 (2020);

Reynolds, RS. Local well-posedness of the topological Euler alignment models of collective behavior, *Nonlinearity*, Volume 33, Number 10, 5176–5214 (2020).

Lear, Reynolds, RS. Global solutions to multi-dimensional topological Euler alignment systems, *Ann. PDE* 8 (2022), no. 1, Paper No. 1, 43 pp.

Fokker-Planck-Alignment model

Locked states are disrupted by stochastic noise

$$\dot{v}_i = \mathrm{s}_i([v]_i - v_i) + \sqrt{2\sigma \mathrm{s}_i}\,\dot{W}_i, \tag{4}$$

where W_i's are independent Brownian motions in $\mathbb{R}^n$. The mean-field limit of solutions satisfies a Fokker-Planck-Alignment equation

$$f_t^\sigma + v \cdot \nabla_x f^\sigma = \sigma \mathrm{s}_\rho \Delta_v f^\sigma + \mathrm{s}_\rho \nabla_v((v - [u^\sigma]_\rho) f^\sigma).$$

So, the expected behavior as $t \to \infty$ would be the same as for the linear Fokker-Planck equation which is a relaxation to the global Maxwellian

$$f^\sigma \to \mu_{\sigma,\bar{u}} = \frac{1}{(2\pi\sigma)^{n/2}} e^{-\frac{|v-\bar{u}|^2}{2\sigma}},$$

where $\bar{u}$ is the mean velocity. If such a convergence holds true, then the alignment of the original system can be recovered in the limit of vanishing noise $\sigma \to 0$:

$$\lim_{\sigma\to 0} \lim_{t\to\infty} f^\sigma(t) = \delta_{v=\bar{u}} \otimes \, \mathrm{d}x.$$

– Duan, Fornasier, and Toscani (2010): relaxation in the Cucker-Smale case

$$f_t + v \cdot \nabla_x f = \sigma \rho_\phi \Delta_v f + \nabla_v((\rho_\phi v - (u\rho)_\phi) f),$$

for perturbation data,

$$f = \mu_{\sigma,\bar{u}} + g\sqrt{\mu_{\sigma,\bar{u}}}, \qquad \|g_0\|_{H^k(\mathbb{T}^n \times \mathbb{R}^n)} \leqslant \varepsilon,$$

for some small $\varepsilon > 0$.

– Choi (2016): relaxation for purely local model

$$f_t + v \cdot \nabla_x f = \sigma \Delta_v f + \nabla_v((v - u) f),$$

in the perturbative settings also.

These results are inspired by techniques from collisional models (Landau, Boltzmann) by Guo, Duan, and others.

Consider IVP for FPA based on Cucker-Smale protocol

$$f_t + v \cdot \nabla_x f = \sigma \rho_\phi \Delta_v f + \nabla_v ((\rho_\phi v - (u\rho)_\phi) f),$$

Theorem (RS, 2022)

Suppose $f_0 \in H^k_l$, $k, l \geqslant n+3$, and suppose $\phi = \psi * \psi$. Then there exists a unique solution to FPA in H^k_l classical global solution to FPA, such that $\rho_- > 0$ uniformly for all $t > t_0 > 0$, and f relaxes to the corresponding Maxwellian at an exponential rate

$$\|f(t) - \mu_{\sigma,\bar{u}}\|_{L^1(\mathbb{T}^n \times \mathbb{R}^n)} \leqslant c_1 \sigma^{-1/2} e^{-c_2 \sigma^{1/2} t},$$

for some c_1 depending on the initial data, and $c_2 > 0$ depending only on the parameters of the system.

Proof consists of several steps:

– global well-posedness in weighted Sobolev spaces;

– uniform gain of positivity, $f \geqslant ae^{-b|v|^2}$, where a, b are time-independent for $t > t_0$.

– estimate on the spectral gap of $[\cdot]_\rho$: $\varepsilon \sim \rho_-^3$. This is where we use $\phi = \psi * \psi$.

– hypocoercivity implied by the uniform spectral gap.

Assuming $\bar{u} = 0$ by Galilean invariance and $\sigma = 1$, consider $h = f/\mu$:

$$\partial_t h = -\rho_\phi A^* A h - Bh + A^*((u\rho)_\phi h),$$

where

$$A = \nabla_v, \quad A^* = v - \nabla_v, \quad B = v \cdot \nabla_x.$$

We have the entropy

$$\mathcal{H} = \int_{\mathbb{T}^n \times \mathbb{R}^n} h \log h \, \mathrm{d}\mu,$$

which obeys two forms of entropy law:

- non-dissipative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\mathcal{I}_{vv}(h) + (u, [u]_\rho)_{\rho\rho_\phi},$$

where

$$\mathcal{I}_{vv}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{|\nabla_v h|^2}{h}\,\mathrm{d}\mu, \qquad (u, [u]_\rho)_{\rho\rho_\phi} = \int_{\Omega^n} (u\rho)_\phi u\rho\,\mathrm{d}x,$$

- non-dissipative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\mathcal{I}_{vv}(h) + (u, [u]_{\rho})_{\rho\rho_{\phi}},$$

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$$\mathcal{I}_{vv}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{|\nabla_v h|^2}{h}\,\mathrm{d}\mu, \qquad (u, [u]_{\rho})_{\rho\rho_{\phi}} = \int_{\Omega^n} (u\rho)_{\phi} u\rho\,\mathrm{d}x,$$

- dissipative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leqslant -(u, u)_{\rho\rho_{\phi}} + (u, [u]_{\rho})_{\rho\rho_{\phi}}.$$

• non-dissipative

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• dissipative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \leqslant -(u, u)_{\rho\rho_\phi} + (u, [u]_\rho)_{\rho\rho_\phi}.$$

We seek to find the spectral gap

$$-(u, u)_{\rho\rho_\phi} + (u, [u]_\rho)_{\rho\rho_\phi} \leqslant -\varepsilon(u, u)_{\rho\rho_\phi}.$$

Suppose for the moment that we control this gap uniformly for $t > t_0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \lesssim -\mathcal{I}_{vv}(h) - (u, u)_{\rho\rho_\phi}.$$

Next: use the full Fischer information

$$\mathcal{I} = \mathcal{I}_{vv} + \varepsilon \mathcal{I}_{xv} + \mathcal{I}_{xx} \gtrsim \mathcal{H} \quad \text{(log-Sobolev inequality)},$$

$$\mathcal{I}_{xv}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{\nabla_x h \cdot \nabla_v h}{h} \, \mathrm{d}\mu, \quad \mathcal{I}_{xx}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{|\nabla_x h|^2}{h} \, \mathrm{d}\mu.$$

Then one computes a la Villani-Desvillettes,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I} \leqslant c_1 \mathcal{I}_{vv} - c_2 \mathcal{I}_{xx} + c_3 (u, u)_{\rho \rho_\phi}.$$

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So,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[c_4 \mathcal{H} + \mathcal{I} \right] \leqslant -c_5 \left[c_4 \mathcal{H} + \mathcal{I} \right].$$

In particular, by the Csiszár-Kullback inequality

$$\|f - \mu\|_1^2 \leqslant \mathcal{H} \leqslant c e^{-ct}.$$

Spectral gap

$$\mathcal{E}_0 - \mathcal{E}_1 \geqslant \varepsilon \mathcal{E}_0.$$

Instead we use the Low Energy Method and look for

$$\mathcal{E}_1 - \mathcal{E}_2 \geqslant \varepsilon \mathcal{E}_1.$$

One can achieve this by using Bochner-positivity of the kernel:

$$(u, [u]_\rho)_{\rho\rho_\phi} = \int_{\Omega^n} (u\rho)_\phi u\rho \, \mathrm{d}x = \int_{\Omega^n} (u\rho)_\psi^2 \, \mathrm{d}x \geqslant 0.$$

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From the formula for $\mathcal{E}_1 - \mathcal{E}_2$ shown before, one gets

$$\varepsilon \geqslant \rho_-^3.$$

Gain of positivity

– Villani, Desvillettes (2000) Space-homogeneous Fokker-Planck;

– Henderson, Snelson, Tarfulea, (2020) Gain of positivity for Boltzmann and Landau;

– F. Anceschi, Y. Zhu (2021) provided a time-dependent gain for general FP equations with bounded drift.

– J. Guerand, C. Imbert (2022) weak Harnack inequality for supersolutions.

Theorem

There exist time-independent constants $a, b > 0$ which depend only on $\mathcal{H}_0$ such that

$$f(t, x, v) \geqslant b e^{-a|v|^2}, \quad \forall x \in \mathbb{T}^n, \ v \in \mathbb{R}^n, \ t > 1. \tag{5}$$

Consequently,

$$\rho_- \geqslant c(a, b).$$

Hence, the spectral gap is uniform in time and previous estimates apply. QED.

THANK YOU!!!