Some recent developments in wave turbulence theory

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THE ABEL SYMPOSIUM
JUNE 13–16, 2023
PARTIAL DIFFERENTIAL EQUATIONS
WAVES, NONLINEARITIES AND NONLOCALITIES

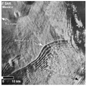
SIMONS FOUNDATION

Wave Turbulence Theory

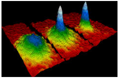
“When in a given physical system a large number of waves are present, the description of each individual wave is neither possible nor relevant. What becomes of physical importance and practical use are the density and the statistics of the interacting waves: this is Wave TurbulenceTheory”

The statistical description of a system of interacting waves is of great importance in physics:

Gravity



Internal waves



Bose Einstein Condensate

The Energy Spectrum

Consider a PDE (dispersive, fluid ...). Let $u(t,x)$

$u : [0,T] \times M \longrightarrow \mathbb{C}, \mathbb{R}$, $M = \mathbb{R}^n, \mathbb{T}^n$ solution.

then $\left\{ |\hat{u}(t,k)|^2 \right\}_k$ is the Energy Spectrum

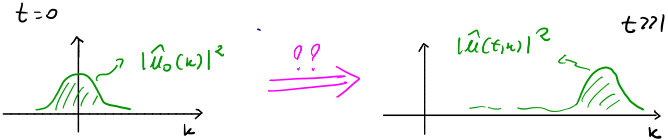
A major question in Wave Turbulence is to understand properties of the Energy Spectrum

Transfer of Energy

Consider the IVP

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \end{cases} \qquad x \in \mathbb{R}^2, \mathbb{T}^2$$

Question of Bourgain:



- Linear case : $\hat{u}(t,k) = e^{it|k|^2}\hat{u}_0(k) \Longrightarrow |\hat{u}(t,k)|^2 = |\hat{u}_0(k)|^2$

Bourgain's Idea : Check $\sum_k |\hat{u}(t,k)|^2 \langle k\rangle^{2s} \xrightarrow[t\to\pm\infty]{} \,?$

- Nonlinear case in $\mathbb{R}^2$: $\exists\, u^{\pm} \in H^s$, $s \geq 2$ s.t.

$\| u(t) - S(t)u^{\pm} \|_{H^s} \xrightarrow[t\to\pm\infty]{} 0 \Longrightarrow \| u(t) \|_{H^s} \leq C.$

- Nonlinear case in $\mathbb{R}$ or $\mathbb{T}$: IVP is integrable system

Conservation laws $\Longrightarrow \| u(t) \|_{H^m} \leq C_m$, $m \in \mathbb{N}$

Wave kinetic equation

Clearly it would be much more effective to derive an effective equation for the spectrum $\{|\hat{u}(t,k)|^2\}_k$. When possible this is called the Wave kinetic Equation (WKE).

(*) Formal Derivation (see Nazarenko's book)

(**) Rigorous Derivation (much harder, more later)

On energy transfer

Energy transfer: bounds from above

Theorem [Bourgain, Sohinger, Planchon-Tzvetkov-Visciglia]
the solution $u(t,x)$ of NLS, $x \in \mathbb{T}^2$

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{s-1+\varepsilon} \qquad \forall\, t,\ \varepsilon>0$$

The proof is based high-low method, upsidedown I-method, integration by parts.

Remarks: If one assumes the torus to be irrational better results are available. See Deng-Germain, Deng in $\mathbb{T}^3$. Also more later.

Energy transfer: is there growth?

Theorem [Colliander-Keel-S-Takaoka-Tao] Consider the IVP

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u \\ u|_{t=0} = u_0 \quad x \in \mathbb{T}^2 \text{ (rational)} \end{cases}$$

Fix $s > 1$, $0 < \sigma \ll 1$, $K \gg 1$. Then $\exists\, u_0 \in H^s$ s.t. $\|u_0\|_{H^s} \leq \sigma$ and $\exists\, T \gg 1$ s.t. $\|u(t)\|_{H^s} \geq K$.

Also holds for irrational numbers "close" to rational

Giuliani - Guardia

Remark: this is a "weak" result since we do not know what happens after time T.

(See also the work of Guardia - Hani - Haus - Maspero - Procesi)

Idea of the proof

Solve for $u(t,x) = \sum_{n\in\mathbb{Z}^2} a_n(t)\, e^{i(t|n|^2 + x\cdot n)}$ (assume $\mathbb{T}^2$ square)

$$\Longrightarrow \quad \boxed{i\partial_t a_n = |a_n|^2 a_n - \sum_{n_1,n_2,n_3\in\Gamma(n)} a_{n_1}\overline{a_{n_2}}\, a_{n_3}} \quad \text{(FNLS)}$$

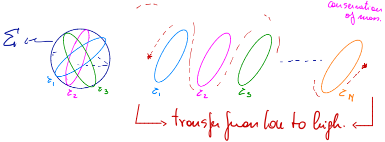
$$\Gamma(n) = \left\{ (n_1, n_2, n_3) \,/\, \begin{array}{l} n = n_1 - n_2 + n_3 \\ \omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0 \end{array} \right\}$$

(n, n_1, n_2, n_3) are in resonance

By making further restrictions on the resonant system becomes:

$$\Rightarrow \boxed{\begin{cases} i\,\dot{b}_j = |b_j|^2 b_j - 2 b_{j-1}^2 \bar{b}_j - 2 b_{j+1}^2 \bar{b}_j \\ b_1(t) = b_N(t) = 0 \qquad j = 1, \dots, N \\ b_j(0) = b_j \end{cases} \quad \text{(Toy Model)}}$$

The Toy Model "lives" on $\Sigma = \{ x \in \mathbb{C}^N \,/\, |x|^2 = 1 \}$ conservation of mass.



Some preliminary remarks

(*) Are these results sharp? No!

(**) What do we expect? Maybe a $\log|t|$ growth for $|t| \gg 1$.

(see results by Bourgain on linear with potential)

Large literature on the topic: Bambusi, Berti, Colliander, Delort, Guardia, Hani, Haus, Maspero, Oh, Procesi, W.M. Wang ....

Hani - Pausader - Tzvetkov - Visciglia:

the cubic, defocusing NLS on $\mathbb{R} \times \mathbb{T}^d$ (rational) for $d = 2, 3, 4$ at $t = \pm\infty$ presents a dynamics dictated by the Toy Model

$$\|u(t_n)\|_{H^s(\mathbb{R} \times \mathbb{T}^d)} \geq \exp\left(C (\log\log t_n)^{\frac{1}{2}}\right)$$

for a sequence $t_n \to \infty$

New Results for irrational tori

This work is joint with A. Hrabski, Y. Pan, B. Wilson

Definition: We say that a torus $\mathbb{T}^2_\alpha$ is irrational if

$$\widehat{\Delta_{\mathbb{T}^2}} = \omega_1^2 k_1^2 + \omega_2^2 k_2^2 \qquad \omega_1^2/\omega_2^2 = \alpha \text{ irrational}$$

Theorem 1: Assume $s \geq 3$ and $u(t,x)$ solution to the cubic, defocusing NLS on $\mathbb{T}^2_\alpha$, α irrational and algebraic, and $u(0,x) = u_0 \in H^s$, $\operatorname{supp} \hat{u}_0 \subseteq B_R$. then $\|u(t)\|_{H^s} \leq C_R\,[1+|t|]$ for $|t| \gg 1$.

Elements of the proof

* Introduction of a "quasi resonant" associated IVP
* Showing that this IVP is g.w.p. in L^2 and showing that it "almost" decouples into 2 1D cubic NLS problems
* Prove a "stability lemma" that allows us to go back to the full NLS problem.

The 4-waves resonant set for the irrational torus

Recall that $\widehat{\Delta_{\mathbb{T}^2}}(k_1,k_2) = \omega_1^2 k_1^2 + \omega_2^2 k_2^2 := \lambda_k$

$$\mathcal{R} = \left\{ (k_1,k_2,k_3,k_4) \;\middle/\; \begin{array}{l} k_1 - k_2 + k_3 - k_4 = 0 \\ \lambda_{k_1} - \lambda_{k_2} + \lambda_{k_3} - \lambda_{k_4} = 0 \end{array} \right\}$$

resonant set

$\Downarrow$ (thanks to the irrationality!)

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$$

$$\mathcal{R}_i := \left\{ (k_1,k_2,k_3,k_4) \;\middle/\; \begin{array}{l} k_1^i + k_3^i = k_2^i + k_4^i \\ (k_1^i)^2 + (k_3^i)^2 = (k_2^i)^2 + (k_4^i)^2 \end{array} \right\}$$

There is a decoupling into two 1D resonant sets!

The 4-waves quasiresonant set

Definition: Fix $\Lambda, \varepsilon > 0$, we define the quasi-resonant set

$$\Omega(\Lambda, \varepsilon) = \left\{ (k_1, k_2, k_3, k_4) \;/\; k_1 + k_3 = k_2 + k_4 \text{ and } \left| \lambda_{k_1} - \lambda_{k_2} + \lambda_{k_3} - \lambda_{k_4} \right| \leq \frac{\Lambda}{\left(|k_1|^2 + |k_2|^2 + |k_3|^2 + |k_4|^2 \right)^{1+\varepsilon}} \right\}$$

Note: The constant Λ is used, when encountering small denominators, to offset the large data assumption. More later.

Theorem 2 Consider the IVP

this is the resonant part of $|v|^2 v$

$$(NLS)^* \begin{cases} i\partial_t v + \Delta v = (|v|^2 v)^* \\ v|_{t=0} = u_0 \end{cases}$$

The $(NLS)^*$ conserves the mass (i.e. $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$)

Moreover $(NLS)^*$ is globally well-posed in $\boxed{L^2(\mathbb{T}^2_\alpha)}$, and

if $\text{supp}\, \widehat{u_0} \subseteq B_R$ then $\exists\, M > 0$ s.t.

$$\|F^{-1}(\chi_{B_M^c} \widehat{v}(t))\|_{H^s} = 0 \qquad \forall\, t \geq 0, \ \forall\, s \geq 0$$

Note: Here M depends on (Λ, ε), on R and on the irrationality of α.

Remarks on theorem 2 :

1) In the C-K-S-T-T work the dynamics of the Toy model was happening inside that of the $(NLS)^*$. Theorem 2 confirms more in details that in the very irrational case there is no growth from the resonant set.

2) Note that global well posedness in L^2 for the full periodic cubic NLS is a major open problem. This is due to the loss of ε-derivative in the Strichartz estimates.

Corollary: Assume u_0 is such that

$$\operatorname{supp} \hat{u}_0 \subseteq B_R$$

then if v is solution to $(NLS)^*$ s.t. $v(0) = u_0$ then

$$\|v(t)\|_{H^s} \leq C \qquad \forall t \in \mathbb{R} \text{ and } \forall s \geq 0$$

Proof: $\exists M$ s.t. $\displaystyle\sum_{|k|>M} \langle k\rangle^{2s} |\hat{v}(t,k)|^2 = 0$, then

$$\sum_{|k|} \langle k\rangle^{2s} |\hat{v}(t,k)|^2 = \sum_{|k|\leq M} \cdots + \sum_{|k|>M} \cdots \leq M^{2s} \sum_{|k|} |\hat{v}(t,k)|^2$$

$$= M^{2s} \|u_0\|_{L^2}^2$$

Ingredients of the proof

- Roth's theorem : This theorem allows us to say that

$$\#\left\{(k_1,k_2,k_3,k_4) \;\middle/\; \overset{k_1+k_3=k_2+k_4}{0<\underset{\uparrow \text{ non resonant}}{\left|\lambda_{k_1}+\lambda_{k_3}-\lambda_{k_2}-\lambda_{k_4}\right|}} < \overset{\text{quasi-resonant}}{\frac{\Delta}{(|k_1|^2+|k_2|^2+|k_4|^2+|k_3|^2)^{1+\varepsilon}}}\right\} \leq \overset{\text{finite}}{C_{\Delta,\alpha,\varepsilon}}$$

- the decoupling of the resonant set $R = R_1 \cup R_2$ into 1D resonant sets. Recall that 1D cubic NLS is globally well-posed in L^2 and it is integrable.

Main Proposition: let $\hat{v}(t,k) =: z_k(t)$ solution to $(NLS)^*$. If

$$N_M^s(z) := \sum_{\substack{k=(m,\ell) \\ |m|>M}} \left[(1+|m|^2)^s + (1+|\ell|^2)^s \right] |z_k|^2 + \sum_{\substack{k=(m,\ell) \\ |\ell|>M}} \left[(1+|m|^2)^s + (1+|\ell|^2)^s \right] |z_k|^2$$

Then $\exists M>0$ s.t. $\frac{d}{dt} N_M^s(z) = 0$ for all t, for all $s \geq 0$

Remark: The proof is based on the 1D splitting of the resonant set and on the fact that $\exists M > 0$ s.t. outside B_M there are only resonant frequencies.

Remarks on the stability lemma

After a rescaling, one needs to prove that the non-quasiresonant part of the solution is "small".

This is equivalent to estimating:

$$\int_0^t \sum_{S_k^c} a_{k_1} \overline{a_{k_2}} a_{k_3}(\tau)\, e^{i\tau\Theta}\, d\tau, \qquad \boxed{\Theta = \lambda_{k_1} - \lambda_{k_2} + \lambda_{k_3} - \lambda_k}$$

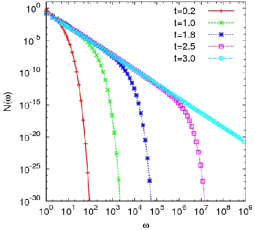
S_k = quasiresonant set.

smallness

Note that in $S_k^c \Longrightarrow |\Theta|^{-1} \leq (\max|k_i|)^{2(\gamma+1)} \frac{1}{\Lambda}$

$\Longrightarrow$ Conclude by integration by parts.

From dispersive equations to wave kinetic equations



Numerical solutions of the isotropic 3-wave kinetic equation –
C. Connaughton

From dispersive equations to wave kinetic equations

Consider the periodic NLS

$$\partial_t u + \Delta u = \underbrace{\varepsilon}_{\text{weak nonlinearity}} |u|^2 u$$

$$u|_{t=0} = u_0 \qquad x \in \mathbb{T}^d_{\underbrace{L}_{\text{size of torus}}}$$

What one wants to study, after assuming an initial distribution for $\widehat{u_0}$, is:

$$\lim_{\varepsilon \to 0, L \to \infty} \mathbb{E}\left(|\hat{u}(\varepsilon^{-2}\tau, k)|^2 \right) =: n_k(\tau)$$

and show that $\boxed{\partial_\tau n_k = Q(n_k)}$ wave kinetic equation

Can we derive the wave kinetic equation?

Fundamental original work on this topic by:
Peierls, Hasselman, Benney-Saffman-Newell, Zakharov, L'vov, Pomeau, Nazarenko, ...

In these works one starts from a certain weakly nonlinear dispersive equation (NLS, KdV, ...) with parameters ε, L and a background probability, then various types of formal approximations and limits are taken

$\Rightarrow$ WKE is obtained!

Example of a formal derivation of a WKE

Consider the Zakharov-Kuznetsov (ZK) equation

$$\partial_t \phi(x,t) = -\Delta \partial_{x_1} \phi(x,t) + \varepsilon\, \partial_{x_1}(\phi^2(x,t)) \qquad x \in [-L,L]^d$$

Let $n_k(t) = \mathbb{E}(|\hat{\phi}(k,t)|^2)$. At the kinetic time $t = \varepsilon^{-2}\tau$

taking $L \to \infty$ then $\varepsilon \to 0$ $\implies$ $\partial_\tau n_k(\tau) = Q(n_k(\tau))$

$$\boxed{\begin{aligned} Q(n_{k_1}) = \int dk_2\, dk_3\, |k_1^1 k_2^1 k_3^1|^2\, \delta(\omega(k_3)+\omega(k_2)-\omega(k_3)) \\ \times\, \delta(k_2+k_3-k_1)\,[n_{k_2} n_{k_3} - n_{k_1} n_{k_2}\, \mathrm{sign}(k_1^1)\, \mathrm{sign}(k_3^1) \\ - n_{k_1} n_{k_3}\, \mathrm{sign}(k_1^1)\, \mathrm{sign}(k_2^1)] \end{aligned}}$$

collision operator

$\omega(k) = k^1 |k|^2$

Define $a_k(t) := \hat{\psi}(t,k) / \sqrt{|k'|}$

Assume $a_k(t)$ are Random Phase Amplitude (RPA) fields. We want to write:

$$a_k(t) = a_k^{(0)}(t) + \varepsilon\, a_k^{(1)}(t) + \varepsilon^2 a_k^{(2)}(t) + \dots$$

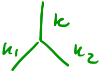
We derive $a_k^{(i)}$ $i = 0, 1, 2$ from the $\widehat{(ZK)}$:

$$\boxed{\dot{a}_k = i\,\omega(k)\, a_k + i\,\varepsilon \sum_{k = k_1 + k_2} \operatorname{sign}(k')\, a_{k_1} a_{k_2}}$$

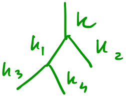
$$a_n^{(0)} = a_n(0) = \hat{\phi}_0(n) \quad \text{(initial datum)}$$

$$a_n^{(1)} = -i\,\mathrm{sign}(k') \sum_{n=n_1+n_2} V_{n_1 n_2 k}\, a_{n_1}^{(0)} a_{n_2}^{(0)} \int_0^t e^{i\omega_{12}^k s}\, ds$$

$$\omega_{12}^n = \omega(n_1) + \omega(n_2) - \omega(n)$$



$$a_n^{(2)} = -2 \sum_{\substack{n=n_1+n_2 \\ n_1 = n_2+n_3}} \mathrm{sign}(k'k_1')\, V_{n n_1 n_2} V_{n_3 n_4 k_4} \,\cdot$$



$$\cdot\, a_{n_2}^{(0)} a_{n_3}^{(0)} a_{n_4}^{(0)} \int_0^t \int_0^s e^{i(\omega'_{34}\sigma + \omega_{12}^n s)}\, d\sigma\, ds$$

Finally one writes $\longrightarrow$ ignore terms with ε^k, $k>2$.

$$n_k(t) = \mathbb{E}(|a_k(t)|^2) \cong \langle (a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)})(a_k^{(0)} + \varepsilon a_k^{(1)} + \varepsilon^2 a_k^{(2)}) \rangle$$

replace the expressions for $a_k^{(i)}$, $i=0,1,2$,

uses RPA and keeping only ε^2 and taking

$L \to \infty$ then $\varepsilon \to 0$

obtain

$$\boxed{\partial_t n_k = Q(n_k)}$$

Mathematical literature: rigorous derivation

- Erdos-Yau, Erdos-Salmhofer-Yau:
  Random $\boxed{\text{linear}}$ Schrödinger on a lattice setting
  → linear Boltzmann (kinetic time) → heat equation (diffusion time $t = \lambda^{-2-\varepsilon}$)

- Lukkarinen-Spohn: Random $\boxed{\text{cubic}}$ NLS at equilibrium and on a lattice setting
  → (linearized) wave kinetic equation at kinetic time.

Random Initial Data:

- Buckmaster-Germain-Hani-Shatah: NLS in continuum case
  $\longrightarrow$ below kinetic time (linear kinetic equation)
- Collot-Germain, Deng-Hani: NLS in continuum case
  $\longrightarrow$ strictly below kinetic time (linear kinetic equation)
- Deng-Hani: NLS in continuum case
  $\longrightarrow$ at kinetic time (nonlinear kinetic equation)

$i\partial_t \phi + \Delta\phi = \lambda |\phi|^2 \phi$, on periodic torus $[0,L]^d$ $d \geq 3$

- Lukkarinen-Vuoksenmaa: NLS in lattice case
  $\longrightarrow$ at kinetic time $d \geq 4$.
- Ma: ZK equation with dissipation and in continuum.
  WKE before kinetic time

Recent work by S.-Tran

We consider the stochastic ZK equation

$$\begin{cases} d\phi(x,t) = -\Delta\partial_x \phi(x,t)\,dt + \varepsilon\partial_{x_1}(\phi^2(x,t))\,dt + \varepsilon^{\theta}\underbrace{\partial_{x_1}\phi \odot dW(t)}_{\text{Stochastic term}} \\ \phi(x,0) = \phi_0(x) \end{cases}$$

$\phi_0(x) \longrightarrow$ randomly distributed

$\varepsilon \ll 1, \quad 0 < \theta \ll 1$

The equation is considered on a lattice

$$\Lambda = \{0, 1, \ldots 2L\}^d$$

$d \geq 2$ (dimension)

L in $\mathbb{N}$.

Passing to frequency space

We write $k = (k^1, \ldots, k^d) \in \Lambda_* = \left\{ -\frac{L}{2L-1}, \ldots, 0, \ldots, \frac{L}{2L-1} \right\}^d$

$$\boxed{\omega_k = \omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1) + \cdots + \sin^2(2\pi k^d)\right]}$$

[dispersive relation]

$\overline{\omega}_k = \sin(2\pi k^1)$

$$W(x,t) = \sum_{k \neq 0} \frac{W_k(t)}{\overline{\omega}(k)} e^{i 2\pi k \cdot x}$$ [Stochastic term]

$\{W_k(t)\}$ = sequence of independent real Wiener processes on (Ω, F, P).

$W_{-k}(t) = -W_k(t) \quad \forall\, k \in \Lambda^* = \Lambda_* \setminus \{0\}$.

Set $a_k = \dfrac{\hat{\psi}(k)}{\sqrt{|\bar{\omega}(k)|}}$ and rewrite the equation

$$\boxed{\begin{aligned} &da_k = i\,\omega(k)\,a_k\,dt + i\,\varepsilon^{\theta} a_k \circ dW_k \\ &i\,\varepsilon \int_{(\Lambda^*)^2} dk_1\,dk_2\ \mathrm{sig}(k^1) \sqrt{|\bar{\omega}(k)\,\bar{\omega}(k_1)\,\bar{\omega}(k_2)|}\ \delta(k-k_1-k_2)\,a_{k_1} a_{k_2}\,dt \end{aligned}}$$

Definition [two points correlation function] density function

$$f(a(t)) = \int |a(t)|^2\, d\rho(t) := \langle a\,\bar{a} \rangle$$

Statement of the main result

Consider the two-points correlation function

$$f(k,t) = \langle a(t,k)\, \bar{a}(t,k) \rangle = \int d\rho\, |a_k(t)|^2$$

Theorem [S.-Tran] Let $d \geq 2$, under suitable (but general) assumptions on the initial distribution ρ_0, if $t = \varepsilon^{-2}\tau$, $\tau \ll 1$

$$\lim_{\varepsilon \to 0, L \to \infty} f(k, \varepsilon^{-2}\tau) = f^{\infty}(k,\tau) \quad \text{and}$$

$$\frac{\partial}{\partial \tau} f^{\infty}(k,\tau) = Q(f^{\infty})(k,\tau) \quad \text{3-Wave Kinetic Equation}$$

The difficulties

- In the rigorous derivation one needs to estimate all the Feynman graphs
- The discreate setting is much more complicated than the continuum setting
- The dispersion relation is very singular
- The quadratic nonlinearity is not as good as the cubic nonlinearity

How we dealt with the obstacles

- We concentrated on the study of the equation for the density function $\rho(t)$ [Liouville equation]
- The stochastic term acts only on angles not magnitude and gives to the Liouville equation some dissipation w.r.t. the angle variables.
- We looked for a weaker type of convergence and this allowed for L and ε not to be coupled.

Thanks for your attention