Low regularity and long time solutions in quasilinear dispersive flows

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This is joint work with Mihaela Ifrim
and also in part with Albert Ai

Nonlinear dispersive problems:

$$i\partial_t u - A(D_x)u = N(u), \qquad u(0) = u_0 \in H^s \qquad \text{(QDE)}$$

Linear characteristic set:

$$\Sigma = \{\tau + a(\xi) = 0\}$$

Group velocity:

$$v_\xi = a'(\xi)$$

Dispersive models:

$$\nabla^2 a(\xi) \neq 0$$

Smooth nonlinearity:

$$N(u) = N(u, \bar{u})$$

May be as strong as A or stronger

Resonant/nonresonant interactions

- relative to the linear A flow
- relative to the linearized flow

Several examples of dispersion relations

- NLS: $a(\xi) = \xi^2$
- KdV: $a(\xi) = \xi^3$
- (Half-) wave: $a(\xi) = |\xi|$
- Deep gravity waves $a(\xi) = |\xi|^{\frac{1}{2}}$
- Capillary waves $a(\xi) = |\xi|^{\frac{3}{2}}$
- Shallow gravity waves $a(\xi) = \sqrt{\xi \tanh \xi}$
- Shallow capillary waves: $a(\xi) = \sqrt{\xi^3 \tanh \xi}$

The nonlinearity

a) Classified by strength:

- semilinear (e.g. NLS3, KdV), Lipschitz dependence on data
- quasilinear (e.g. water waves), continuous dependence on data

b) Classified by leading homogeneity:

- quadratic,

$$N(u) = Q_1(u,u) + Q_2(u,\bar{u}) + Q_3(\bar{u},\bar{u})$$

- cubic, e.g.

$$N(u) = Q(u,\bar{u},u)$$

- higher order

c) Classified by leading order nonlinear effect (cubic case):

- defocusing
- focusing

Two questions

1. Local well-posedness: Is the evolution (QDE) locally well-posed in H^s ?

2. Global well-posedness: Are there global dispersive solutions for (QDE) for small initial data in H^s ?

Quasilinear local well-posedness

[Enhanced] Hadamard local well-posedness in Sobolev spaces

$$u(0) \in H^s$$

- existence of solutions u in the class $C(0, T; H^s)$
- uniqueness of solutions, either directly for rough solutions, or as unique limits of smooth solutions
- continuous dependence in H^s, i.e. continuity of the data to solution map
$$H^s \ni u(0) \to u \in C(0, T; H^s)$$
- weak Lipschitz dependence, i.e. for two H^s solutions u and v we have the difference bound
$$\|u - v\|_{C(0,T;L^2)} \lesssim \|u(0) - v(0)\|_{L^2}$$
- higher regularity

Local well-posedness

What is s ?

- Classically true for large enough s by energy estimates
  - for the full equation in H^s
  - for the linearized equation in H^{s_0} for some $s_0 < s$.
- Scaling exponent s_c gives universal threshold.
- Aspirational goal: bring s as close as possible to s_c.

Nonlinear wave interactions:

- Strength of the nonlinearity (also related to scaling)
- Resonant versus nonresonant interactions, null conditions
- Role of dispersion
  - Linear dispersive decay (e.g. Strichartz)
  - Multilinear estimates (e.g. parallel vs. transversal interactions)

Making good choices:

- Good variables (Alinhac)
- Good quasilinear energies

Global well-posedness

1. Classical: Conserved energy + LWP $\Rightarrow$ GWP
   - no dispersive decay information

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2. Modern: Strichartz + energy ⇒ GWP + scattering
   - requires higher order nonlinearity or higher dimension, e.g. for cubic nonlinearity $d \geq 4$ (NLW) or $d \geq 3$ (NLS) [subcritical d]

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3. Contemporary:
   Small, smooth and localized data $\Rightarrow$ GWP with dispersive decay
   - vector field methods
   - scattering vs. modified scattering

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   Small, smooth and localized data ⇒ GWP with dispersive decay
   - vector field methods
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4. Ongoing research:
   Small, ~~smooth~~ and ~~localized~~ data ⇒ GWP with dispersive decay
   - focusing vs defocusing (cubic case)
   - no vector field methods
   - weaker notion of scattering

Bony's paradifferential formalism (expanded)

Full nonlinear equation:

$$u_t + N(u) = 0$$

Linearized equation:

$$v_t + DN(u)v = 0$$

Linear paradifferential equation:

$$w_t + T_{DN(u)}w = 0$$

Full equation in paradifferential formulation

$$u_t + T_{DN(u)}u = R(u)$$

Linearized equation in paradifferential formulation

$$v_t + T_{DN(u)}v = R_{lin}(u)v$$

LWP key ideas

1. Energy estimates (Kato)

$$\frac{d}{dt}E^s(u) \lesssim B E^s(u)$$

   - $E^s(u)$ quasilinear energy
   - B control parameter, e.g. $B \approx \|u\|_{H^s}$; better $B \approx \|u\|_{W^{k,\infty}}$.
   - Also for the linearized equation in weaker topology

2. Strichartz estimates

$$\|u\|_{L^p_t W^{k,\infty}} \lesssim \|u_0\|_{H^s}$$

   - Allows one to estimate control parameter in energy estimates.
   - Work with paradifferential equation, perturbative source term
   - Main difficulty: variable coefficients, with low regularity

3. Bootstrap argument to combine energy + Strichartz
   - best captured using Tao's frequency envelopes, see Ifrim-T. primer

Paradifferential Strichartz estimates

- Easy for constant coefficients
  - stationary phase methods

- Valid for C^2 coefficients
  - wave packet parametrices Smith, T.
  - Oscillatory integral parametrices

- Strichartz estimates with loss of derivatives (T., Bahouri-Chemin)
  - For each frequency λ, find "semiclassical" time scales $\delta t \approx \lambda^{-\delta}$ where loss-less Strichartz holds.
  - Add these bounds to get Strichartz with deriv. losses on unit time.

- Strichartz estimates without loss of derivatives
  - Construct more accurate, rough wave packet parametrices
  - NLW (Smith-T. '05), 2D gravity waves (Ai '18), 1D NLS (Ifrim-T. '23)
  - use the fact that the coefficients solve an equation (Klainerman)

Energy estimates, take 1

- Quadratic energy estimates (Kato)

$$\frac{d}{dt} E^s(u) \lesssim B E^s(u)$$

- Cubic energy estimates for problems with quadratic nonlinearities and null structure

$$\frac{d}{dt} E^{s,3}(u) \lesssim A B E^{s,3}(u)$$

where

$$A \approx \|u\|_{W^{k_0,\infty}} \text{ (at scaling)}, \qquad B \approx \|u\|_{W^{k_1,\infty}} \text{ (above scaling)}$$

  - modified energy method Hunter-Ifrim-T. '12-'14
  - quasilinear adaptation of the normal form method
  - scale invariant bound
  - Also *para-diagonalization method* of Alazard-Delort '13
  - very useful for long time bounds. less for low regularity LWP

Energy estimates, take 2

- Cubic energy estimates for problems with quadratic nonlinearities and null structure

$$\frac{d}{dt} E^{s,3}(u) \lesssim AB E^{s,3}(u)$$

$$A \approx \|u\|_{W^{k_0,\infty}} \text{ (at scaling)}, \qquad B \approx \|u\|_{W^{k_1,\infty}} \text{ (above scaling)}$$

- Balanced cubic energy estimates for problems with quadratic/cubic nonlinearities and null structure

$$\frac{d}{dt} E^{s,3}_{bal}(u) \lesssim A^2_{1/2} E^{s,3}_{bal}(u)$$

where

$$A_{1/2} \approx \|u\|_{W^{k_{1/2},\infty}}, \qquad k_{1/2} = \frac{k_0 + k_1}{2}$$

  - introduced by Ai-Ifrim-T. '19 for 2D gravity waves
  - proved for hyperbolic minimal surface equation by Ai-Ifrim-T. '21.
  - very useful for low regularity well-posedness
  - also for low regularity long time dynamics

Gravity waves 2-D

Variables: $\alpha + W(\alpha) =$ surface param, $Q=$ velocity potential on top.
Holomorphic coordinates: $W = PW$, $Q = PQ$ (negative freq)
Differentiated variables:

$$\mathbf{W} = W_\alpha \text{ (slope)}, \qquad R = \frac{Q_\alpha}{1 + W_\alpha} \text{ (complex velocity)}$$

Differentiated equation:

$$\begin{cases} (\partial_t + b\partial_\alpha)\mathbf{W} + \dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}} R_\alpha = G(\mathbf{W}, R) \\ (\partial_t + b\partial_\alpha) R - i\dfrac{(1+a)\mathbf{W}}{1+\mathbf{W}} = K(\mathbf{W}, R) \end{cases} \tag{DWW}$$

where

$$b = 2\Re P\left[\frac{R}{1+\mathbf{W}}\right], \qquad a = 2\Im P[R\bar{R}_\alpha]$$

Taylor coefficient: $a \geq 0$, necessary for well-posedness.

2D Gravity waves at low regularity

$$\begin{cases} (\partial_t + b\partial_\alpha)\mathbf{W} + \dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}} R_\alpha = G(\mathbf{W}, R) \\ (\partial_t + b\partial_\alpha)R - i\dfrac{(1+a)\mathbf{W}}{1+\mathbf{W}} = K(\mathbf{W}, R) \end{cases} \tag{DWW}$$

Sobolev spaces:

$$(\mathbf{W}, R) \in \mathcal{H}^s := H^s \times H^{s+\frac{1}{2}}$$

Critical Sobolev index:

$$s_c = \frac{1}{2}$$

Theorem (Albert Ai-Mihaela Ifrim- DT.'19)

The 2D gravity waves flow is locally well-posed in $\mathcal{H}^s$ for $s \geq \frac{3}{4}$.

Progression of LWP results: gravity waves 2D

		$s - s_c$
Wu ('97)	Energy	4
Alazard-Burq-Zuily ('12)	Energy	$\frac{1}{2} + \epsilon$
Hunter-Ifrim-Tataru ('14)	Cubic energy	$\frac{1}{2}$
Alazard-Burq-Zuily ('14)	Energy + Strichartz w. loss	$\frac{1}{2} - \frac{1}{24} + \epsilon$
Ai ('18)	Energy + sharp Strichartz	$\frac{3}{8} + \epsilon$
Ai-Ifrim-Tataru ('19)	Balanced cubic energy	$\frac{1}{4}$
(in progress) ('21-23)	Balanced cubic energy + Strichartz	$\frac{1}{8}$ (?)

Hyperbolic minimal surfaces: nonlinear waves with null condition

- A time-like submanifold $\Sigma \subseteq \mathbb{R}^{n+2}$ of Minkowski space, critical point of

$$\int_\Sigma dA$$

- Euler-Lagrange equation:

$$-\frac{\partial}{\partial t}\left(\frac{u_t}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)+\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1-u_t^2+|\nabla_x u|^2}}\right)=0$$

- Re-express using trace of Minkowski metric on Σ:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u=0, \qquad g_{\alpha\beta}=m_{\alpha\beta}+\partial_\alpha u\partial_\beta u$$

- aka. Born-Infeld in electromag., aka. zero mean curvature flow, aka. relativistic membrane equation, aka. branes in string theory

Starting point: LWP for generic NLW

Theorem (Smith-T. '01)

Nonlinear wave equations are locally well-posed in $\mathcal{H}^s = H^s \times H^{s-1}$ *for*

$$s > s_c + \frac{1}{2}, \qquad n \geq 3$$

$$s > s_c + \frac{3}{4}, \qquad n = 2$$

- independent work by Klainerman-Rodnianski in the special case of Einstein equations
- sharp result generically by Lindblad's counterexample

Conjecture (T. '02)

The above result can be improved for NLW which satisfy a nonlinear null condition.

LWP for nonlinear waves with null condition

Theorem (Albert Ai, Mihaela Ifrim, D.T. '21)

The time-like minimal surface equation is locally well-posed in $\mathcal{H}^s = H^s \times H^{s-1}$ *for*

$$s \geq s_c + \frac{1}{4}, \qquad n \geq 3$$

$$s \geq s_c + \frac{3}{8}, \qquad n = 2$$

- improves the sharp generic result of Smith-T. by $1/4$ derivatives if $n \geq 3$, and by $3/8$ derivatives if $n = 2$.
- First result proving the null condition LWP conjecture.
- Prior ϵ-removal results by Klainerman-Rodnianski-Szeftel (GR) and Ettinger (minimal surface)

Progression of results: Nonlinear wave equation 3D

		$s - s_c$
Hughes-Kato-Marsden ('76)	Energy	$1 + \epsilon$
Bahouri-Chemin ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{7}{10}$
Tataru ('98-'99)	Energy + Strichartz w. loss	$\frac{3}{4}$ to $\frac{2}{3}$
Klainerman-Rodnianski ('00)	Energy + Strichartz w. loss	$\frac{3-\sqrt{3}}{2}$
Smith-Tataru ('01)	Energy + sharp Strichartz	$\frac{1}{2} + \epsilon$
Kl.-Rod.-Szeftel ('15) [GR]	Energy + sharp Strichartz	$\frac{1}{2}$
Ai-Ifrim-Tataru ('21) [*null*]	Balanced cubic energy + Strichartz w. loss	$\frac{1}{4} + \epsilon$

Key steps

1. Balanced energy bounds for the paradifferential equation
   - first in L^2 by modified energies
   - then in all H^σ by paraconjugation argument
2. Balanced energy bounds for full equation in all H^σ
   - variable coefficient iterated normal form correction
3. Balanced energy bounds for the linearized equation in H^{σ_0}.
   - variable coefficient iterated normal form correction
   - but loss of symmetry in the linearization
4. Small data lossless SE $\rightarrow$ large data SE with losses
   - difficulty: limited use of scaling in inhomogeneous Sobolev spaces
5. Extensive bootstrap argument
   - Combines balanced energies with Strichartz for a family of regularized solutions

GWP conjectures for 1D cubic problems

GWP conjectures for 1D cubic problems

Conjecture (Non-localized data defocusing global well-posedness, Ifrim-T. '22)

1D dispersive problems with a cubic nonlinearity which is conservative and defocusing and with small data have global dispersive solutions.

GWP conjectures for 1D cubic problems

Conjecture (Non-localized data defocusing global well-posedness, Ifrim-T. '22)

1D dispersive problems with a cubic nonlinearity which is conservative and defocusing and with small data have global dispersive solutions.

Conjecture (Non-localized data (focusing) long time well-posedness conjecture, Ifrim-T. '22)

1D dispersive problems with cubic nonlinearity which is conservative and with ϵ-small data have long time ϵ^{-8} dispersive solutions.

Heuristics: trilinear wave packet interactions

$$i\partial_t u - A(D_x)u = N(u)$$

Phase rotation symmetry: $u \to e^{i\theta}u$.

Cubic expansion with phase rotation symmetry:

$$i\partial_t u - A(D_x)u = C(u, \bar{u}, u) + N^{nr}(u)$$

Amplitude equation for $(\xi, \xi, \xi) \to \xi$ interactions:

$$i\dot{A} = c(\xi, \xi, \xi)A|A|^2,$$

always nonperturbative on large time scales.

Here $c(\xi, \xi, \xi) \in \mathbb{R}$ prevents blow-up (exponential growth).

Two assumptions on the symbol of C:

1. Conservative: $c(\xi, \xi, \xi), \nabla c(\xi, \xi, \xi) \in \mathbb{R}$
   $\to$ Wave packet interactions do not increase energy

2. Focusing vs. defocusing:
   $\to$ determines what happens when wave packets get remodulated
   $\to$ determined by the sign of $c(\xi, \xi, \xi)$ vs sthe sign of a''.

A semilinear result (defocusing case)

Theorem (Ifrim-T. '22)

$$i\partial_t u + \Delta u = C(u, \bar{u}, u), \qquad u(0) = u_0$$

Suppose the nonlinearity C is cubic, conservative and defocusing. Then for small initial data

$$\|u_0\|_{L^2} \leq \epsilon \ll 1$$

there exists a unique global solution u so that

$$\|u\|_{L^\infty L^2} \lesssim \epsilon \qquad \textit{(Energy)}$$

$$\|u(t)\|_{L^6} \lesssim \epsilon^{\frac{2}{3}} \qquad \textit{(Strichartz)}$$

$$\|P_A u P_B u\|_{L^2} \lesssim d(v_A, v_B)^{-\frac{1}{2}} \epsilon^2 \qquad \textit{(bilinear } L^2\textit{)}$$

- First result of this type
- no energy conservation is assumed
- global dispersive bounds are obtained
- work in progress: general dispersion relations

A special case: defocusing $NLS^3(\mathbb{R})$

$$i\partial_t + \Delta u = u|u|^2$$

- Globally well-posed in L^2.
- Completely integrable $\Rightarrow$ Conserved energies

Theorem

L^2 solutions satisfy the Strichartz bound

$$\|u\|_{L^6} \lesssim \|u_0\|_{L^2}$$

and the bilinear L^2 bound

$$\|\partial_x |u|^2\|_{cL^2 + \dot{H}^{-\frac{1}{2}}} \lesssim \|u_0\|_{L^2}^2, \qquad c = \|u_0\|_{L^2}$$

Earlier dispersive bounds for H^1 solutions by by Planchon-Vega.

A semilinear result (focusing case)

Theorem (Ifrim-T. '22)

$$i\partial_t u + \Delta u = C(u, \bar{u}, u), \qquad u(0) = u_0$$

Suppose the nonlinearity C is cubic and conservative. Then for small initial data

$$\|u_0\|_{L^2} \leq \epsilon \ll 1$$

there exists a solution u in $[0, \epsilon^{-8}]$ so that

$$\|u\|_{L^\infty[0,\epsilon^{-8};L^2]} \lesssim \epsilon \qquad \textit{(Energy)}$$

and also on ϵ^{-6} time intervals we have:

$$\|u(t)\|_{L^6} \lesssim \epsilon^{\frac{2}{3}} \qquad \textit{(Strichartz)}$$

$$\|P_A u P_B u\|_{L^2} \lesssim d(v_A, v_B)^{-\frac{1}{2}} \epsilon^2 \qquad \textit{(bilinear } L^2\textit{)}$$

- Sharp result, because of the existence of small solitons.

A quasilinear Schrödinger model

$$\begin{cases} iu_t + g(u)\partial_x^2 u = N(u, \partial_x u), \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \\ u(0,x) = u_0(x) \end{cases} \tag{QNLS}$$

- $g = g(u, \bar{u})$ smooth, real valued, $g(0) = 1$.
- $N = N(u, \bar{u}, \partial u, \partial \bar{u})$ is smooth, complex valued, at most quadratic in ∂u.

$$\begin{cases} iu_t + g(u, \partial_x u)\partial_x^2 u = N(u, \partial_x u), \quad u : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \\ u(0,x) = u_0(x) \end{cases} \tag{DQNLS}$$

Sharp local well-posedness

Theorem (Ifrim-T. '23)

a) The cubic (QNLS) *is locally well-posed in* H^s *for* $s > 1$, *and the solutions satisfy*

1. *Uniform* H^s *bounds*
2. *Loss-less Strichartz estimates*
3. *Transversal bilinear* L^2 *bounds.*

b) The same result holds for the cubic (DQNLS) *for* $s > 2$.

- Scaling index $s_c = \frac{1}{2}$ (resp. $s_c = \frac{3}{2}$)
- Regular solutions with localized data Kenig-Ponce-Vega '04
- Rough solutions with $s > 2$ (resp $s > 3$) Marzuola-Metcalfe-T. '14
- Should be generically ill-posed below H^1 (resp. H^2):
  - comparison with NLS^3 below L^2.

Other remarks:

- difference between quadratic and cubic problems (Mizohata, Doi)
- difference between small and large data, nontrapping [KPV], [MMT]

Defocusing global well-posedness

Theorem (Ifrim-T. '23)

a) Consider the cubic (QNLS) *with phase rotation symmetry, conservative and defocusing. Let* $s > 1$. *Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u *which satisfies*

1. *Uniform* H^s *bounds*
2. *Strichartz estimates with* $1/6$ *derivative loss.*
3. *Transversal bilinear* L^2 *bounds (loss-less).*

- First proof of the defocusing GWP conjecture in a quasilinear setting.
- Sharp result in terms of regularity
- Global in time integrated decay bounds ("scattering")

Focusing long time well-posedness

Theorem (Ifrim-T. '23)

a) Consider the cubic (QNLS) *with phase rotation symmetry, and conservative. Let* $s > 1$*. Then for small initial data*

$$\|u_0\|_{H^s} \leq \epsilon \ll 1$$

there exists a unique global solution u *in* $[0, \epsilon^{-8}]$ *which satisfies*

1. *Uniform* H^s *bounds*
2. *Strichartz estimates with* $1/6$ *derivative loss on* ϵ^{-6} *time scale*
3. *Transversal bilinear* L^2 *bounds (loss-less) on* ϵ^{-6} *timescale.*

- First quasilinear proof of the focusing long time WP conjecture.
- Sharp result in terms of regularity.
- Sharp result in terms of time scales (small solitons).

Five key ideas

1. Bootstrap argument via frequency envelopes
   - associated to a dyadic frequency decomposition

2. Energy estimates via density flux identities.
   - carried out in a nonlocal setting, where both the densities and the fluxes involve translation invariant multilinear forms.

3. Modified energies, akin to the I-method.
   - we implement this at the level of density-flux identities, rather than for energy functionals

4. Interaction Morawetz bounds.
   - extended to the setting and language of nonlocal multilinear forms.

5. Strichartz estimates.
   - via wave packet parametrices, after peeling off perturbative errors

The Littlewood-Paley decomposition

Dichotomy for multilinear forms:

- parallel interactions $\longrightarrow$ rely on L^6 Strichartz
- transverse interactions $\longrightarrow$ rely on bilinear L^2

Dyadic frequency decomposition:

$$u = \sum_{\lambda \in 2^{\mathbb{N}}} u_\lambda,$$

size of LP regions dictated by the Hamilton flow.

Goal:

- estimate each u_λ separately
- estimate bilinear interactions

A collection of related equations

Full equation:

$$iu_t + g(u)\partial_x^2 u = N(u, \partial_x u). \tag{QNLS}$$

Linearized equation:

$$iv_t + g(u)\partial_x^2 v = N^{lin}(u)v. \tag{QNLS-lin}$$

Paradifferential equation:

$$iw_{\lambda t} + \partial_x g(u_{<\lambda})\partial_x w_\lambda = f_\lambda \tag{QNLS-para}$$

Full equation in paradifferential form, long time analysis

$$iu_{\lambda t} + \partial_x g(u_{<\lambda})\partial_x^2 u_\lambda = N_\lambda^{nr}(u, \partial_x u) + C_\lambda(u, \bar{u}, u) \tag{QNLS}$$

Linearized equation in paradifferential form

$$iv_{\lambda t} + \partial_x g(u_{<\lambda})\partial_x^2 v_\lambda = N_\lambda^{lin}(u)v. \tag{QNLS-lin}$$

Frequency envelopes

-introduced by Tao to track the time evolution of dyadic energies

- Start with frequency envelope $\{c_\lambda\} \in \ell^2$ for the initial data

$$\|u_{0\lambda}\|_{H^s} \lesssim \epsilon c_\lambda$$

- Show that similar bounds carry over to solutions
- Key assumption on c: *slowly varying*, to control nonlinear leakage.

$$\frac{c_\lambda}{c_\mu} \leq \left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right)^\delta .$$

Bootstrap hypothesis:

$$(BOOT1) \qquad \|u_\lambda\|_{L^\infty L^2} \lesssim C\epsilon c_\lambda \lambda^{-s}$$

$$(BOOT2) \qquad \|u_\lambda(t)\|_{L^6} \lesssim C\epsilon c_\lambda \lambda^{-s}$$

$$(BOOT3) \qquad \|\partial_x(u_\lambda \bar{u}^h_\mu)\|_{L^2} \lesssim C^2\epsilon^2(\lambda+\mu)^{\frac{1}{2}} c_\lambda c_\mu \lambda^{-s}\mu^{-s}(1+\lambda h)$$

- bootstraping both Strichartz and bilinear: Ifrim-T., Benjamin-Ono

Conservation laws in density flux form

- Integral laws in linear/nonlinear case:

$$\mathbf{M} = \int |u|^2 \, dx, \qquad \frac{d}{dt}\mathbf{M} = \int C_m^4(u, \bar{u}, u, \bar{u}) \, dx$$

- Well chosen mass/momentum densities

$$\mathbf{M} = \int M(u, \bar{u}) \, dx, \qquad \mathbf{P} = \int P(u, \bar{u}) \, dx$$

- Density flux identities in linear/nonlinear case:

$$\partial_t M(u, \bar{u}) = \partial_x [g P(u, \bar{u})] + C_m^4(u, \bar{u}, u, \bar{u})$$

$$\partial_t P(u, \bar{u}) = \partial_x [g E(u, \bar{u})] + C_p^4(u, \bar{u}, u, \bar{u})$$

- Frequency localized density-flux identities:

$$\partial_t M_\lambda(u, \bar{u}) = \partial_x [g_{<\lambda} P_\lambda(u, \bar{u})] + C_{m,\lambda}^4(u, \bar{u}, u, \bar{u})$$

$$\partial_t P_\lambda(u, \bar{u}) = \partial_x [g_{<\lambda} E_\lambda(u, \bar{u})] + C_{p,\lambda}^4(u, \bar{u}, u, \bar{u})$$

Energy corrections for long time results

♣ second generation I-method: correct energies for better conservation (I-team:=Colliander-Keel-Stafillani-Takaoka-Tao)

♡ better strategy: correct densities and fluxes

- Quartic energy correction

$$M_\lambda^\sharp(u,\bar{u}) = M_\lambda(u,\bar{u}) + B^4_{\lambda,m}(u,\bar{u},u,\bar{u}),$$

$$P_\lambda^\sharp(u,\bar{u}) = P_\lambda(u,\bar{u}) + B^4_{\lambda,p}(u,\bar{u},u,\bar{u}),$$

- Density-flux identities:

$$\partial_t M_\lambda^\sharp = \partial_x(P_\lambda + R^4_{\lambda,m}) + F^{4,nr}_{\lambda,m} + R^6_{\lambda,m}$$

$$\partial_t P_\lambda^\sharp = \partial_x(E_\lambda + R^4_{\lambda,p}) + F^{4,nr}_{\lambda,p} + R^6_{\lambda,p}$$

  - This requires solving a nontrivial division problem,

$$c^4 = \Delta^4 \xi^2 \cdot b^4 + \Delta^4 \xi \cdot r^4 + (\xi_{odd} - \xi_{even})^2 q^{4,nr}$$

  - Energy bounds follow by direct integration

Bilinear L^2 estimates

- cannot use linear theory, as (i) problem is quasilinear and (ii) nonlinearity is nonperturbative
- Nonlinear idea: Interaction Morawetz

- introduced by I-team '03 for 3D NLS
- one dimensional version by Planchon-Vega

Baby version: $u, v \geq 0$ densities

$$\partial_t u = \partial_x f, \qquad \text{moves to the left } f > 0$$

$$\partial_t v = \partial_x g, \qquad \text{moves to the right } g < 0$$

Interaction functional:

$$I(u, v) = \int_{x<y} u(x)v(y)\, dx dt$$

$$\frac{dI}{dt} = \int_{\mathbb{R}} fv - ug\, dx > 0 \qquad \text{(transversality bound)}$$

Dispersive Interaction Morawetz

"momentum is moving to the right faster than the mass"

1. Interaction Morawetz functional, diagonal case:

$$I(u_\lambda, u_\lambda) = \int_{x<y} M_\lambda^\sharp(x) P_\lambda^\sharp(y) - M_\lambda^\sharp(y) P_\lambda^\sharp(x)\, dxdy$$

Time differentiation:

$$\frac{d}{dt} I(u_\lambda, u_\lambda) \approx \|\partial_x(u_\lambda \bar{u}_\lambda)\|_{L^2}^2 + \|u_\lambda\|_{L^6}^6 + \text{Errors (6,8,10)}$$

- used to prove the L^6 Strichartz and diagonal bilinear L^2.

2. Transversal Interaction Morawetz functional:

$$I(u_\lambda, u_\mu) = \int_{x<y} M_\lambda^\sharp(x) P_\mu^\sharp(y) - M_\mu^\sharp(y) P_\lambda^\sharp(x)\, dxdy$$

Time differentiation:

$$\frac{d}{dt} I(u_\lambda, u_\mu) \approx \|\partial_x(u_\lambda \bar{u}_\mu)\|_{L^2}^2 + \text{Errors (6,8,10)}$$

- used to prove the off-diagonal bilinear L^2 bound.

Lossless Strichartz estimates

Established at the level of the paradifferential equation:

$$i w_{\lambda t} + \partial_x g(u_{<\lambda}) \partial_x w_\lambda = f_\lambda, \qquad w(0) = w_0 \qquad \text{(QNLS-para)}$$

Main challenge: variable coefficient problem

- SE with derivative loss: from sharp SE on semiclassical time scales (Staffilani-T. '02, Burq-Gerard-Tzvetkov '06, etc.)
- SE without loss on asymptotically flat spaces (Robbiano-Zuily '06, Hassell-Tao-Wunsch '06, T. 07)

All the above require at least C^2 coefficients. Here, $g - 1 \in L^\infty H^{1+}$!

Key ideas:

- flatten metric with change of coordinates
- use equation for u
- allow for a large class of source terms f_λ
- use bilinear L^2 estimates to peel off rough parts of coefficients.
- use wave packet parametrix (Marzuola-Metcalfe-T.)

Summary

A common circle of ideas for both low reqularity well-posedness and long time/global solutions in quasilinear dispersive pde's:

1. Modified energy methods for problems with null condition:
   - Modified energy method $\Longrightarrow$ cubic energies for long time solutions
   - Balanced cubic energies $\Longrightarrow$ low regularity LWP
   - Low reg. Strichartz using equation for coeff $\Longrightarrow$ LWP

2. Multilinear interaction methods in cubic 1D flows
   - Defocusing GWP conjecture
   - focusing long time well-posedness conjecture
   - proved for both semilinear and quasilinear Schrödinger flows
   - density flux identities $\Longrightarrow$ interaction estimates
   - global solutions for nonlocalized data at LWP regularity

Thank you for your attention !

Linear dispersion in 1D

1. Fundamental solution:

$$K(t,x) \approx \frac{1}{\sqrt{ta''(\xi_v)}} e^{it\phi(v)}, \qquad v = x/t$$

$$a'(\xi_v) = v, \qquad \phi'(v) = \xi_v \qquad \text{(Legendre)}$$

A1: $t^{-\frac{1}{2}}$ decay (for localized or L^1 data)

2. Translation invariant bounds:

$$\|e^{itA}u_0\|_S \lesssim \|u_0\|_{L^2} \qquad \text{(Strichartz)}$$

$$\swarrow \downarrow \searrow$$

$$L^\infty L^2 \quad L^6 \quad L^4 L^\infty$$

$$\|u_A u_B\|_{L^2} \lesssim |v_A - v_B|^{-\frac{1}{2}} \|u_{A0}\|_{L^2} \|u_{B0}\|_{L^2} \qquad (\text{ bilinear } L^2)$$

A2: L^6 + transversal L^2 bounds (for L^2 data)