The Quartic Integrability and Long Time Existence of Water Waves in 2d

Sijue Wu

University of Michigan, Ann Arbor, USA

Partially supported by National Science Foundation

The gravity water wave equations

We assume that

- the air density is 0, the fluid density is 1. $(0, -g)$ is the gravity.
- the fluid is inviscid, incompressible, irrotational,
- the surface tension is zero.

Let $\Omega(t)$ be the fluid domain, $\partial\Omega(t)$ be the interface at time t.

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{0}, -g) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \partial\Omega(t) \\ (1, \mathbf{v}) \text{ is tangent to } (t, \partial\Omega(t)) \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

When surface tension is zero, the motion can be subject to the Taylor instability

- Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq 0; \tag{2}$$

on the interface $\partial\Omega(t)$. $\mathbf{n}$ is the unit outward normal to the fluid domain $\Omega(t)$.

- G. I. Taylor (1949)
- Strong Taylor sign condition:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0. \tag{3}$$

History

Newton, Stokes, Levi-Civita, G.I.Taylor.....

Local wellposedness in Sobolev spaces

- Nalimov (1974), Yoshihara (1982), W. Craig (1985): 2D, local wellposedness for small Sobolev data;
- S. Wu (1997, 99): 2D and 3D, Local wellposedness for arbitrary data in Sobolev spaces H^s, for $s \geq 4$.
  Proved that the strong Taylor sign condition always holds, i.e.
  $$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{4}$$
  for $C^{1,\gamma}$, $\gamma > 0$ interfaces.
- Christodoulou & Lindblad (2000), Iguchi(2001), Ogawa & Tani (2002), Ambrose & Masmoudi(2005), D. Lannes (2005), Lindblad (2005), Coutand & Shkoller (2007), P. Zhang & Z. Zhang (2007), Shatah & Zeng (2008)
  Local wellposedness with additional effects: nonzero surface tension, finite depth, nonzero vorticity, assuming the strong Taylor sign condition holds.

Global behavior for small, smooth and sufficiently localized data: – Gravity-WW

- S. Wu (2009), S. Wu (2011): almost global for 2-D, global well-posedness for 3-D;
- Germain, Masmoudi & Shatah (2012): global well-posedness for 3-D
- Ionescu & Pusateri (2015), Alazard & Delort (2015): global existence and modified scattering for 2-D;
- Hunter, Ifrim & Tataru (2016), Ifrim & Tataru (2016), Ai, Ifrim & Tataru (2019, 2020): lowered the regularity threshold;

- main ideas: 1. after suitable change of variables, the water wave equation has no quadratic nonlinear terms; 2. use the dispersive decay property for sufficiently localized solutions to get extended lifespan.
- If the data is smooth and of size ϵ, and non-localized, the solution exists on time of order $O(\epsilon^{-2})$.
- The main mechanism is that there are no 3-waves resonant interactions.

Question:

Is it possible to further remove the cubic nonlinearities of the gravity-WW using a normal form transformation?

The Hamiltonian point of view, the periodic case

- Zakharov (1968): formulated 2d water wave equations as a Hamiltonian system
- Dyachenko & Zakharov (1994): no 3-wave interactions, all 4-wave interaction coefficients vanish on the non-trivial resonant manifold
- Craig & Wolfolk (1995): formal derivation of the Birkhoff normal form transformation of order 4
- Craig & Sulem (2016): studied mapping properties
- Berti, Feola & Pusateri (2018): existence for time of order $O(\epsilon^{-3})$ for periodic small, smooth data of size ϵ.

Dyachenko & Zakharov (1994):

- No 3-wave resonant interactions;
- After the Birkhoff normal form procedure, the Hamiltonian

$$H \circ \Phi = H^{(2)} + H^{(4)} + \dots .$$

- There are two types of 4-wave resonant interactions:
  - the trivial ones,
  - the Benjamin-Feir resonances,
  - there is no cancelations on the trivial ones,
  - the coefficients in $H^{(4)}$ vanish on the Benjamin-Feir resonances.
- Berti, Feola, Pusateri (2018): bounded, invertible Birkhoff normal forms; non-symplectic, non-explicit;
- For the whole line case, one also need to deal with near resonances.

- The computations of Dyachenko & Zakharov, Craig & Wolfolk, Berti, Feola & Pusateri are all carried out in the Fourier space using tools such as the Birkhoff normal forms from dynamical system.
- Question: how does this fact manifest, explicitly and naturally, in the physical space?

How do we solve the water wave equation (1)?

- A usual approach is to reduce from (1) to an equation on the interface, and study the interface equation.
- Recover $\mathbf{v}$ from its value on the interface $\partial\Omega(t)$ by solving $\Delta \mathbf{v} = 0, \quad \text{in } \Omega(t)$;
- We consider the 2d case. Use the Riemann mapping variable.
- We describe the approach in Wu (1997, 2009, 2018)

The surface equation in Lagrangian coordinates

- We identify $(x, y) = x + iy$;
- The free surface
$$\partial\Omega(t) : z = z(\alpha, t),$$
$\alpha \in \mathbb{R}$; α is the Lagrangian coordinate.
- $z_t = z_t(\alpha, t)$ velocity, $z_{tt} = z_{tt}(\alpha, t)$ acceleration,
- the gravity $(0, -g) = (0, -1) = -i$.
- $-\nabla P = -\frac{\partial P}{\partial \mathbf{n}}\mathbf{n} := i\mathfrak{a}z_\alpha$,
$\mathbf{n} = \frac{iz_\alpha}{|z_\alpha|}$, $\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}}\frac{1}{|z_\alpha|}$;
- $\bar{z}_t$ boundary value of the holomorphic function $\bar{\mathbf{v}}$.

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{0}, -1) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div} \mathbf{v} = 0, \qquad \operatorname{curl} \mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \partial\Omega(t) \end{cases} \tag{5}$$

Equation of the free surface:

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \bar{z}_t = \mathfrak{H} \bar{z}_t \end{cases} \tag{6}$$

where $\mathfrak{H}$ is the Hilbert transform,

$$\mathfrak{H} f(\alpha) = \frac{1}{\pi i} \int \frac{z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)} f(\beta)\, d\beta$$

The surface equation in the Riemann mapping framework

Let

$$\Psi = \Psi(\cdot, t) : P_- \to \Omega(t)$$

be the Riemann mapping satisfying $\lim_{z' \to \infty} \Psi_{z'}(z', t) = 1$; here P_- is the lower half plane.

Let

- $Z(\alpha'; t) := \Psi(\alpha'; t)$ – interface;
- $Z(h(\alpha, t), t) = z(\alpha, t)$, $b = h_t \circ h^{-1}$;
- $h(\alpha, t) = \alpha'$;
- $Z_t := D_t Z = v(Z(\alpha'; t); t)$ – velocity;
- $Z_{tt} := D_t Z_t$ – acceleration;
- $D_t := \partial_t + b\partial_{\alpha'}$ – the material derivative;
- $\partial_{\alpha'} Z := Z_{,\alpha'}$, $Z_{tt,\alpha'} = \partial_{\alpha'}\{Z_{tt}\}$, etc.

Surface equation in Riemann mapping coordinate

Equation of the free surface:

$$\begin{cases} Z_{tt} + i = \dfrac{iA_1}{\bar{Z}_{,\alpha'}}, \qquad (Z_{tt} = D_t Z_t = (\partial_t + b\partial_\alpha)Z_t) \\ (I - \mathbb{H})\bar{Z}_t = 0, \quad (I - \mathbb{H})\left(\dfrac{1}{Z_{,\alpha'}} - 1\right) = 0 \end{cases} \tag{7}$$

$$A_1 = 1 - \operatorname{Im}[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} \geq 1,$$
$$b := h_t \circ h^{-1} = \operatorname{Re}(I - \mathbb{H})\frac{Z_t}{Z_{,\alpha'}}. \tag{8}$$

- Z_t - velocity, $\frac{1}{Z_{,\alpha'}}$ - fluid domain,
- $-|Z_{,\alpha'}| \frac{\partial P}{\partial \mathbf{n}} = A_1 \geq 1;$

$$\mathbb{H}f(\alpha') = \frac{1}{\pi i}\int \frac{1}{\alpha' - \beta'} f(\beta')\, d\beta'.$$

The quasilinear equation in Riemann mapping coordinate:

- Taking one time derivative (material derivative) to (6), we get a quasilinear equation:

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha)\bar{z}_t = -i\mathfrak{a}_t\bar{z}_\alpha \quad \left(= \frac{\mathfrak{a}_t}{\mathfrak{a}}(\bar{z}_{tt} - i)\right).$$

- In Riemann mapping variable it is:

$$\left(D_t^2 + i\frac{A_1}{|Z_{,\alpha'}|^2}\partial_{\alpha'}\right)\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \tag{9}$$

where

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = quadratic \tag{10}$$

- $$\mathcal{P}\bar{Z}_t = \text{quadratic, lower order terms.}$$

The normal form transformation in Wu (2009)

- Let ϕ be the velocity potential, i.e. $\mathbf{v} = \nabla\phi$,
- Let $\psi = \phi(z(\alpha, t), t)$
- $\Lambda = (I - \mathfrak{H})\psi$, $\Pi = (I - \mathfrak{H})(z - \bar{z})$ satisfy equations:

$$(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Lambda = cubic, \qquad (\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi = cubic,$$

- a coordinate change given by

$$\kappa = 2\Re z - h$$

removes the quadratic nonlinearities on the left hand side:

$$(\partial_t^2 - i\partial_\alpha)(\Lambda \circ \kappa^{-1}) = cubic, \qquad (\partial_t^2 - i\partial_\alpha)(\Pi \circ \kappa^{-1}) = cubic.$$

- The basic energy functional: for θ holomorphic,

$$E(\theta, t) = \int \frac{1}{\mathfrak{a}}|\partial_t\theta|^2 + i\partial_\alpha\theta\bar{\theta}\, d\alpha.$$

- $$\frac{d}{dt}E(\partial^j \Lambda, t) = \text{quartic}, \qquad \text{same for } \partial^j \Pi.$$
- Existence of solution for time of order $O(\epsilon^{-2})$ for data of size ϵ.
- A further use of the method of vector fields yields the almost global existence result in Wu (2009).
- Similar quartic estimate played key roles in Ionescu & Pusateri (2015), Alazard & Delort (2015), Hunter, Ifrim & Tataru (2016)......;
- The mechanism behind is the absence of 3-wave interactions.

Question:

Is it possible to construct a sequence of energy functionals $\mathfrak{E}_j(t)$, so that

$$\frac{d}{dt}\mathfrak{E}_j(t) = \text{quintic?}$$

- The construction works for both the whole line and the periodic cases.

Results on the structure of the water wave equation

- Begin with reconstructing a quartic energy functional sequence.
- Define: $\mathbb{P}_H = \frac{1}{2}(I + \mathbb{H})$, $\mathbb{P}_A = \frac{1}{2}(I - \mathbb{H})$
- Begin with

$$Q := (I + \mathbb{H})(\psi \circ h^{-1}),$$

$$\big(D_t \mathbb{P}_H D_t + i \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}\big) Q = i \, \mathbb{P}_A\big(Z_t\big(1 - \frac{1}{Z_{,\alpha'}}\big)\big) + \bar{Z}_t\big(\frac{1}{\bar{Z}_{,\alpha'}} - 1\big)\big). \quad (11)$$

- Let

$$\Theta^{(0)} := Q, \qquad \Theta^{(j)} := (\mathbb{P}_H D_t)^j Q, \quad (12)$$

$$\big(D_t \mathbb{P}_H D_t + i \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}\big) \Theta^{(j)} =: G^{(j)}. \quad (13)$$

- $\mathbb{P}_H G^{(0)} = 0$.
- Derive a formula for $\mathbb{P}_H(G^{(j)})$, – it is cubic with symmetric structures.

Proposition (energy identity)

Let Θ_1, Θ_2 be holomorphic, Define

$$E(t) = \Re\Big(\int i\,\partial_{\alpha'}\Theta_2\overline{\mathbb{P}_H D_t\Theta_1}\,d\alpha' - \int i\,\partial_{\alpha'}\Theta_1\overline{\mathbb{P}_H D_t\Theta_2}\,d\alpha'\Big). \tag{14}$$

Then

$$\frac{d}{dt}E(t) = \Re\Big(\int i\,\partial_{\alpha'}\Theta_2\overline{(\mathbb{P}_H G_1)}\,d\alpha' - \int i\,\partial_{\alpha'}\Theta_1\overline{(\mathbb{P}_H G_2)}\,d\alpha'\Big). \tag{15}$$

where $G_k := (D_t\mathbb{P}_H D_t + i\,\frac{1}{|Z_{,\alpha'}|^2}\partial_{\alpha'})\Theta_k$, *for* $k = 1, 2$.

- Let

$$E_j(t) = \Re\Big(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\Theta^{(j+1)}}\,d\alpha' - \int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\Theta^{(j+2)}}\,d\alpha'\Big). \quad (16)$$

- From (15),

$$\frac{d}{dt}E_j(t) = \Re\Big(\int i\,\partial_{\alpha'}\Theta^{(j+1)}\overline{\mathbb{P}_H G^{(j)}}\,d\alpha' - \int i\,\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_H G^{(j+1)}}\,d\alpha'\Big);$$

$$\begin{aligned} &2\big(\mathbb{P}_H G^{(l+1)} - \mathbb{P}_H D_t \mathbb{P}_H G^{(l)}\big) \\ &= \mathbb{P}_H\Big(\frac{1}{\bar{\bar{Z}}_{,\alpha'}}\Big(< \bar{Z}_t, i\,\frac{1}{\bar{\bar{Z}}_{,\alpha'}}, D_{\alpha'}\Theta^{(l)} > + < -i\,\frac{1}{Z_{,\alpha'}}, Z_t, D_{\alpha'}\Theta^{(l)} > \Big)\Big), \end{aligned}$$

- $< f, g, h >= \frac{1}{\pi i}\int \frac{(f(\alpha')-f(\beta'))(g(\alpha')-g(\beta'))(h(\alpha')-h(\beta'))}{(\alpha'-\beta')^2}\,d\beta'.$
- $\frac{d}{dt}E_j(t)$ is quartic in terms of only the derivatives of $\frac{1}{Z_{,\alpha'}}$ and Z_t;

Removing the quartic terms

- $$-i\,\frac{1}{Z_{,\alpha'}}+i=\bar{Z}_{tt}+quadratic,\quad D_{\alpha'}\Theta^{(k)}=D_t^k\bar{Z}_t+quadratic$$

- Derive an energy identity which moves the D_t derivatives from factors to factors:
- Provided $\mathcal{P}f$, $\mathcal{P}g$, $\mathcal{P}h$, $\mathcal{P}q$ are quadratic,

$$\begin{aligned}&\frac{d}{dt}\iint\frac{\bar{f}\,\mathfrak{D}_t(g\,\bar{h}\,q)-(\mathfrak{D}_t\bar{f})\,g\,\bar{h}\,q}{(\alpha'-\beta')^2}\,d\alpha'd\beta'\\&=2\iint\frac{\bar{f}\,\mathfrak{D}_t(g\,\bar{h})\,\mathfrak{D}_tq+\bar{f}\,\mathfrak{D}_t(g\,\mathfrak{D}_t\bar{h})\,q}{(\alpha'-\beta')^2}\,d\alpha'd\beta'+quintic,\end{aligned}\tag{17}$$

- Let $\theta = \bar{Z}_t(\alpha', t) - \bar{Z}_t(\beta', t)$,
- let $\mathfrak{D}_t = \partial_t + b(\alpha', t)\partial_{\alpha'} + b(\beta', t)\partial_{\beta'}$,

$$C_{2,j} = \frac{1}{4\pi}\sum_{k=0}^{j-1}(-1)^k \iint \frac{(D_t^j Z_t \mathfrak{D}_t - D_t^{j+1} Z_t)\theta\, \mathfrak{D}_t^k \bar{\theta}\, \mathfrak{D}_t^{j-k-1}\theta}{(\alpha' - \beta')^2}\, d\beta'\, d\alpha'$$

$$+ \frac{1}{4\pi}(-1)^j \iint D_t^j Z_t \frac{\theta\, \mathfrak{D}_t^j \bar{\theta}\ \theta}{(\alpha' - \beta')^2}\, d\beta'\, d\alpha'. \tag{18}$$

$$
\begin{aligned}
C_{1,j} &= \frac{1}{2\pi}\sum_{l=0}^{j-1}\sum_{k=0}^{l}\iint \frac{(D_t^j Z_t \mathfrak{D}_t - D_t^{j+1} Z_t)\mathfrak{D}_t^{l-k}(\mathfrak{D}_t^k\theta\ \bar{\theta}\,\mathfrak{D}_t^{j-l-1}\theta)}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha' \\
&+ \frac{1}{4\pi}\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^k\iint \frac{(D_t^j Z_t \mathfrak{D}_t - D_t^{j+1} Z_t)\mathfrak{D}_t^{1+l}\theta\,\mathfrak{D}_t^k\bar{\theta}\,\mathfrak{D}_t^{j-l-2-k}\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha' \\
&- \frac{1}{8\pi}\sum_{l=0}^{j-2}\sum_{k=0}^{j-l-2}(-1)^k\iint \frac{(\theta\mathfrak{D}_t - \mathfrak{D}_t\theta)\mathfrak{D}_t^{j-l-1}\bar{\theta}\,\mathfrak{D}_t^{j-k-1}\theta\,\mathfrak{D}_t^{k+l+1}\bar{\theta}}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha' \\
&+ \frac{1}{2\pi}\sum_{l=0}^{j-1}\iint D_t^j Z_t\,\frac{\mathfrak{D}_t^{j-l-1}\theta\ \bar{\theta}\,\mathfrak{D}_t^{1+l}\theta}{(\alpha'-\beta')^2}\,d\beta'\,d\alpha',
\end{aligned}
\tag{19}
$$

Main result on the quartic integrability of WWE

Theorem

Let

$$\mathfrak{E}_j(t) = E_j(t) - \Re\Big(\int i\partial_{\alpha'}\Theta^{(j)}\overline{\mathbb{P}_H G^{(j)}}\,d\alpha' + C_{1,j}(t) + C_{2,j}(t)\Big). \tag{20}$$

Then

$$\frac{d}{dt}\mathfrak{E}_j(t) = quintic \tag{21}$$

with desirable structures.

- $\mathfrak{E}_j(t)$ controls only the spatial derivatives of $\frac{1}{Z_{,\alpha'}}$ and Z_t, waves with large steepness and velocity can be small in $\mathfrak{E}_j(t)$ for $j \geq 2$;
- $\frac{d}{dt}\mathfrak{E}_j(t)$ is quintic in terms of the spatial derivatives of $\frac{1}{Z_{,\alpha'}}$ and Z_t.

Scaling law

- If $(\bar{Z}_t, Z)$ is a solution of (7)-(8), then

$$(\bar{Z}_t^\lambda, Z^\lambda) := (\lambda^{-1/2}\bar{Z}_t(\lambda\alpha', \lambda^{1/2}t), \lambda^{-1}Z(\lambda\alpha', \lambda^{1/2}t)) \qquad (22)$$

  is also a solution of (7)-(8).
- Scaling invariant norms:

$$\|\frac{1}{Z_{,\alpha'}} - 1\|_{\dot{H}^{1/2}(\mathbb{R})}, \quad \|\bar{Z}_{t,\alpha'}\|_{L^2(\mathbb{R})},$$

$$\|\frac{1}{Z_{,\alpha'}} - 1\|_{L^\infty(\mathbb{R})}.$$

- Let

$$L(t) = \|\frac{1}{Z_{,\alpha'}}(t)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\bar{Z}_{t,\alpha'}(t)\|_{L^2(\mathbb{R})} + \|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(t)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\partial_{\alpha'}^2 \bar{Z}_t(t)\|_{L^2(\mathbb{R})}. \quad (23)$$

Theorem

1. Let $J \geq 2$. Assume that the initial data $(\bar{Z}_t(0), \frac{1}{Z_{,\alpha'}}(0)) \in \cap_{\frac{1}{2}\leq s\leq J} \dot{H}^s(\mathbb{R}) \times \dot{H}^{s-\frac{1}{2}}(\mathbb{R})$. Then there are constants $m_0 > 0$, and $\varepsilon_0 > 0$, such that for all $0 < \varepsilon \leq \varepsilon_0$, if the data satisfies

$$L(0) \leq \varepsilon, \qquad \|\frac{1}{Z_{,\alpha'}}(0) - 1\|_{L^\infty} < 1, \qquad E_1(0)E_3(0) \leq m_0^2, \quad (24)$$

then the lifespan of the unique classical solution for the 2d water wave equation (1) is at least of order $O(\varepsilon^{-3})$. During this time, the solution is as regular as the initial data and $L(t)$ remains small.

Theorem (continued)

If instead of (24) *the data satisfies*

$$\begin{aligned} \|\frac{1}{Z_{,\alpha'}}(0)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\bar{Z}_{t,\alpha'}(0)\|_{L^2(\mathbb{R})} \leq \varepsilon, \\ \|\frac{1}{Z_{,\alpha'}}(0) - 1\|_{L^\infty} < 1, \qquad E_1(0)E_3(0) \leq m_0^2, \end{aligned} \tag{25}$$

then the lifespan of the unique classical solution for the 2d water wave equation (1) *is at least of order* $O(\varepsilon^{-5/2})$*. During this time, the solution is as regular as the initial data.*

Remark: $\|\frac{1}{Z_{,\alpha'}}(0)\|_{\dot{H}^{1/2}(\mathbb{R})} + \|\bar{Z}_{t,\alpha'}(0)\|_{L^2(\mathbb{R})}$ is scaling invariant.

Remarks

- $E_1(t) \approx \|\frac{1}{Z_{,\alpha'}}(t) - 1\|^2_{L^2(\mathbb{R})} + \|\bar{Z}_t(t)\|^2_{\dot{H}^{1/2}(\mathbb{R})}$,
- $E_3(t) \approx \|\frac{1}{Z_{,\alpha'}}(t)\|^2_{\dot{H}^1(\mathbb{R})} + \|\bar{Z}_{t,\alpha'}(t)\|^2_{\dot{H}^{1/2}(\mathbb{R})}$,
- $E_1(t)E_3(t)$ is scaling invariant,
- Sobolev embedding: $\|\frac{1}{Z_{,\alpha'}}(t) - 1\|^2_{L^\infty} \leq c_0 E_1(t)E_3(t)$,
- $E_1(t)E_3(t)$ remains controlled for time of order $O(\varepsilon^{-3})$,
- If $\|\frac{1}{Z_{,\alpha'}}(0) - 1\|_{L^\infty} \leq 1 - 2\delta$, then $\|\frac{1}{Z_{,\alpha'}}(t) - 1\|_{L^\infty} \leq 1 - \delta$ for time of order $O(\varepsilon^{-3})$.
- $m_0 > 0$ need not be small, $0 < \delta < 1$ is arbitrary.

Remarks

- Part 2 of the Theorem is a consequence of part 1 by a scaling argument.
- The rescaled data $(\bar{Z}_t^\varepsilon(0), Z^\varepsilon(0))$ satisfies

$$\|\partial_{\alpha'}\frac{1}{Z^\varepsilon_{,\alpha'}}(0)\|_{\dot{H}^{1/2}}+\|\partial^2_{\alpha'}\bar{Z}^\varepsilon_t(0)\|_{L^2}=\varepsilon\|\partial_{\alpha'}\frac{1}{Z_{,\alpha'}}(0)\|_{\dot{H}^{1/2}}+\varepsilon\|\partial^2_{\alpha'}\bar{Z}_t(0)\|_{L^2}.$$

- The rescaled solution $(\bar{Z}_t^\varepsilon(t), Z^\varepsilon(t))$ has lifespan of order $O(\varepsilon^{-3})$ implies that the solution $(\bar{Z}_t(t), Z(t))$ has lifespan of order $O(\varepsilon^{-5/2})$.
- The lifespan of the solution is in fact of order $O(\varepsilon^{-3+\frac{1}{2J-2}})$, under the assumption of part 2.
- The interface $Z = Z(\alpha', t)$ is a graph during the lifespan of the solution.

Main ideas

- Use the quintic energy functionals $\mathfrak{E}_j(t)$,
-
$$\mathfrak{E}_j(t) \approx \|\frac{1}{Z_{,\alpha'}}(t) - 1\|^2_{\dot{H}^{\frac{j-1}{2}}(\mathbb{R})} + \|\bar{Z}_t(t)\|^2_{\dot{H}^{\frac{j}{2}}(\mathbb{R})}$$
-
$$\frac{d}{dt}\mathfrak{E}_j(t) = O(\epsilon^5), \qquad \text{for } \; 2 \le j \le 4$$
$$\frac{d}{dt}(\mathfrak{E}_1(t)\mathfrak{E}_3(t)) = O(\epsilon^3)\mathfrak{E}_1(t)\mathfrak{E}_3(t),$$
- provided
$$L(t) \le \epsilon, \qquad \|\frac{1}{Z_{,\alpha'}}(t) - 1\|_{L^\infty} \le 1 - \delta;$$

$0 < \epsilon \le \epsilon_0(\delta)$.

Some further Remarks:

- Similar calculations apply to a variety of free boundary problems, and for some we can derive higher order integrability, of quartic or cubic orders...
- A recent work of Deng, Ionescu, Pusateri (2022): quartic integrability, and longer time existence by using dispersive property.

Thank you!