Talks KGTFA

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The Chern connection in time-frequency analysis

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To the memory of Steve Zelditch (1943-2022)

The Chern connection ∇_{h^n} of the line bundle (L^n, h^n) over a compact Kähler manifold M is the unique connection compatible with the Hermitian metric h^n and the holomorphic structure of M. One main goal of this talk, in the context of the current workshop, is to promote some discussion on how some ideas from Kähler geometry have a natural role in time-frequency analysis. Thus, in the first part, with the caveat imposed by my restricted knowledge of the topic, I will try to present some concepts of Kähler geometry.

One perspective is the understanding of the proper Hilbert subspaces of $L^2(\mathbb{C}^m)$ generated by Gabor transforms of $L^2(\mathbb{R}^m)$, as universal scaling limits of of the space $H^0(M, L^n)$ of global holomorphic sections of the *nth* tensor power of L. A good source of intuition for complex and time-frequency analysts is the remark of Lev and Ortega-Cerdá [17], where it is pointed out that such global holomorphic sections behave similary as members in the Fock space of entire functions (the Gabor space with a Gaussian window). This intuition is sustained by a universality property: for any Kähler manifold M, the universal scaling limit of the Bergman kernel $K_n(z, w)$ of $H^0(M, L^n)$ is the Fock space kernel $K_{\mathcal{F}}(z, w) = e^{z\overline{w}}$ [6]. In the process, the Chern connection $\nabla_{h^n} = \nabla'_{h^n} + \nabla''_{h^n}$ of the line bundle (L^n, h^n), approaches the Chern connection $\nabla_{h_{\mathcal{F}}} = \nabla'_{h_{\mathcal{F}}} + \nabla''_{h_{\mathcal{F}}}$ of the trivial bundle $(L = \mathbb{C} \times \mathbb{C}^m, h_{\mathcal{F}})$, where $h_{\mathcal{F}}(z) = e^{-|z|^2}$. We will see that the resulting operators

$$\nabla'_{h_{\mathcal{F}}} = \partial_z - \overline{z}, \qquad \nabla''_{h_{\mathcal{F}}} = \partial_{\overline{z}}.$$

have a fundamental role in time-frequency analysis. They act on a Gabor transform with Hermite window by raising and lowering the index of the Hermite function. Another important role of these operators is found in the physics of the Quantum-Hall effect, since the Landau operator with a constant magnetic field acting on $L^2(\mathbb{C}, e^{-|z|^2})$, can be written as

$$\mathcal{L}_{z,\overline{z}} := -\partial_z \partial_{\overline{z}} + \overline{z}\partial_{\overline{z}} = -\nabla'_{h_{\mathcal{F}}} \nabla''_{h_{\mathcal{F}}},$$

and $\nabla'_{h_{\mathcal{F}}}$ and $\nabla''_{h_{\mathcal{F}}}$, coincide with the raising and lowering operators of the eigenspaces associated with the integer values (Landau Levels) of the pure, infinitely degenerated, spectrum of $\mathcal{L}_{z,\overline{z}}$ [1].
We will present applications of these ideas in the study of zero correlations of Gaussian random functions in different Landau levels, white noise spectrograms [4, 12, 15] and high-resolution time-frequency analysis. This last part of the talk will center around results obtained with Shirai [3], strongly inspired by the need of understanding algorithms, experimental and heuristic work of Flandrin [13, 18] and of Daubechies, Wang and Wu [7], NuHAG experiments of Hans Feichtinger, and by Feng's study of correlations between zeros and Chern critical points of random analytic functions on Kähler manifolds [9].

The talk is dedicated to the memory of Steve Zelditch, whose work (and of his students and collaborators) had a profound influence on my recent research and in the circle of ideas that I will describe. This started, most notably, with the serendipitious recognition that Flandrin's honeycomb prediction of the ratio of 1/3 between averages of zeros and local maxima of white noise spectrograms [14, 13], was proved for compact Kähler manifolds in [8] and the arguments extended to the study of value averages in [10, 11]. Not surprisingly, Flandrin's prediction, which can be identified with the zeros and maxima in the universal Fock space (Gaussian Entire Functions [5]), could be proved with simplified variations of the compact case. Surprisingly enough, this well studied and more simple universal case had not been considered before and was object of an independent publication [1].

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R. Albesiano

A degeneration approach to Skoda's division theorem

Fix holomorphic functions $h_1, \ldots, h_r$ on $\mathbb{C}^n$. Given another holomorphic function g, is it possible to find holomorphic functions $f_1, ..., f_r$ such that $g = h_1 f_1 + ... + h_r f_r$? In 1972, H. Skoda proved a theorem addressing this question and giving L^2 bounds on the minimal-L2-norm solution. I will sketch a new proof of a Skoda-type theorem, inspired by a degeneration argument of B. Berndtsson and L. Lempert. In particular, we will see how to obtain L^2 bounds on the solution f1, . . . , fr with minimal L^2 norm by deforming a weight on the space of all linear combinations $v_1 f_1 + \ldots + v_r f_r$ to single out the linear combination h1 f1 + . . . + hr fr we are interested in.

B. Berndtsson

Supercurrents in differential and convex geometry.

The formalism of superforms and supercurrents is an attempt to use methods from Kahler geometry in real geometry. I will give a survey over some of its uses, including minimal surfaces, Alexandrov-Fenchel inequalities and valuations on convex bodies.

T. Darvas

The trace operator of quasi-plurisubharmonic functions on compact Kähler manifolds

We introduce the trace operator for quasi-plurisubharmonic functions on compact Kahler manifolds, allowing to study the singularities of such functions along submanifolds where their generic Lelong numbers vanish. Using this construction we obtain novel Ohsawa-Takegoshi extension theorems and give applications to restricted volumes of big line bundles (joint work with Mingchen Xia).

M. Gjertsen

Gabor Analysis and its Symplectic Structure

I will give a basic introduction to Gabor analysis with an emphasis on the role played by symplectic structures. This ties in to work I am currently doing with

Franz Luef, where we use symplectic covariance to introduce an equivalence relation on the set of lattices in $\mathbb{R}^{2d}$ such that equivalent lattices share identical structures of Gabor frames. These equivalence classes are parameterized by symplectic forms on $\mathbb{R}^{2d}$, giving us explicit parameters controlling the behavior of multivariate Gabor frames. We have also obtained a converse to symplectic covariance, characterizing symplectic transformations as the structure preserving transformations of the time-frequency plane in Gabor analysis and hence deepening the link between Gabor analysis and symplectic geometry.

J. Hultgren

On the SYZ-conjecture for hypersurfaces in toric Fano manifolds

I will talk about with the SYZ-conjecture for families of hypersurfaces in toric Fano manifolds. Recent work by Yang Li reduces a weak version of the SYZ-conjecture in this setting to the solvability of a real Monge-Ampère equation on the boundary of a polytope. Curiously, this Monge-Ampère equation is solvable for some families and not solvable for some families. I will explain how solvability can be described in terms of properties of the minimizers of an optimal transport problem, and how another subtle aspect of the PDE, the presence of a free boundary, becomes less mysterious when viewed through the lens of optimal transport. Finally, I will highlight some of the many open problems related to this. This is based on joint work with Rolf Andreasson, Thibaut Delcroix, Mattias Jonsson, Enrica Mazzon and Nick McCleerey.

Z. Mouayn

Coherent states attached to Landau levels on the complex projective space $\mathbb{CP}^n$

We are dealing with magnetic Laplacians on the complex projective n-space. We are interested in spaces of their bounded eigenfunctions, whose reproducing kernels are linked to coherent states. We also attach to each of these eigenspaces a Berezin transform. The heat equation and heat coefficients associated with these operators are also discussed.

T. Nguyen

A HILBERT BUNDLES APPROACH TO THE SHARP STRONG OPENNESS THEOREM

This is joint work with Xu Wang. We show how to use a Hörmander-type of result together with complex Brunn-Minkowski theory to obtain a general monotonicity theorem (on weighted norms of holomorphic sections associated with certain Hilbert bundles). As an applica- tion of the latter, we also give a generalization of Guan's sharp strong openness theorem.

J. Scheffler

Auxiliary Monge-Ampère Equations on Orbifolds

During the last years, the technique developped by Guo-Phong-Tong of using auxiliary Monge-Ampère equations on compact Kähler manifolds led to estimates for the Green's function, diameter estimates and a Sobolev type inequality. It is of utmost importance – e.g. for applications to singular spaces – that the constants are uniform even if the Kähler metrics degenerate; in particular, no bounds for the Ricci curvature are needed. At the heart of the arguments is a mean-value inequality by Guo-Phong-Sturm which we could generalize to the singular setting of Kähler orbifolds (cf. arXiv:2404.02812). I will lecture on this technique including a presentation of my own result and a flavor of the arguments.

Gabor frame bounds optimizations

Irina Shafkulovska

We denote with $\pi(\lambda)f(t) = e^{2\pi i\omega\cdot t}f(t-x)$ the time-frequency shift of the function $f \in L^2(\mathbb{R})$ by $\lambda = (x,\omega) \in \mathbb{R}^{2d}$. We define the *Gabor system* of the window $g \in L^2(\mathbb{R}^d)$ along $\Lambda \subseteq \mathbb{R}^{2d}$ as the collection of time-frequency shifts of g given by

$$\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}. \tag{1}$$

We call a Gabor system a *Gabor frame* if it is a frame for $L^2(\mathbb{R}^d)$, that is, if there exist positive constants $0 < A \leq B < \infty$ such that

$$A\,\|f\|_{L^2}^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\,\|f\|_{L^2}^2\,. \tag{2}$$

If the chain of inequalities (2) holds, we call the optimal A and B the lower and upper frame bound, respectively. Furthermore, then there exists a sequence $(h_\lambda)_{\lambda\in\Lambda} \subseteq L^2(\mathbb{R}^d)$, called the *dual frame* of $\mathcal{G}(g,\Lambda)$, such that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle h_\lambda = \sum_{\lambda\in\Lambda} \langle f, h_\lambda\rangle g_\lambda, \quad f \in L^2(\mathbb{R}^d). \tag{3}$$

The *condition number* of $\mathcal{G}(g,\Lambda)$ is the ratio $\kappa = B/A$.

In this talk, we discuss the importance of the condition number for the convergence rate of the reconstruction of f from the measurements $\langle f, \pi(\lambda)g\rangle_{\lambda\in\Lambda}$. We look into a few well-known window functions and families of separable lattices of integer density $n \in \mathbb{N}$, i.e.,

$$\mathfrak{F}_\Lambda(g,n) = \{a\mathbb{Z}\times b\mathbb{Z} : \mathcal{G}(g, a\mathbb{Z}\times b\mathbb{Z}) \text{ is a frame, } (ab)^{-1} = n\}. \tag{4}$$

For the chosen windows, we present unique solutions to the optimization problems

$$\underset{(ab)^{-1}=n}{\arg\max}\, A(a\mathbb{Z}\times b\mathbb{Z}), \quad \underset{(ab)^{-1}=n}{\arg\min}\, B(a\mathbb{Z}\times b\mathbb{Z}), \quad \underset{(ab)^{-1}=n}{\arg\min}\, \kappa(a\mathbb{Z}\times b\mathbb{Z}). \tag{5}$$

The talk is based on joint work with Markus Faulhuber (University of Vienna).

A. Trusiani
TBA

D. Varolin
TBA

X. Wu
Compact Kähler Threefolds with Nef Anticanonical Line Bundles

This presentation elucidates recent collaborative research conducted with Shin-ichi Matsumura, focusing on the characterization of compact Kähler threefolds with nef anticanonical line bundles. The endeavor to classify compact Kähler manifolds, particularly those exhibiting positive tangent bundles or anticanonical line bundles, constitutes a significant pursuit within mathematical discourse. While notable progress has been achieved, particularly through the seminal work of Cao and Höring in delineating the classification of projective manifolds with nef anticanonical line bundles, the analogous task in the compact Kähler setting presents formidable challenges. In dimension three, based on Kähler MMP developped by Höring-Peternell, a classification can be given using analytic tools like Segre currents, Q-conic bundle, orbifold vector bundle, etc.

Y. Xiong
Variational methods to extremal problems in p-Bergman spaces.

The p-Bergman space consists of L^p integrable holomorphic functions. As the classical Bergman theory, we can define the p-Bergman kernel via an extremal problem. Bo-Yong Chen and Liyou Zhang applied the variational method to this extremal problem and established a reproducing formula in p-Bergman spaces. It turns out that the reproducing formula can be used to obtain the $C^{1,1}_{\text{loc}}$ interior regularity of p-Bergman kernels when $p \geq 1$. Moreover, since the (L^2) Bergman kernel gives the optimal constant in the single-point Ohsawa-Takegoshi extension theorem, we may also consider a similar extremal problem for the general case. The variational method will lead to another proof of Berndtsson-Păun's L^p Ohsawa-Takegoshi type extension theorem. The talk is based on some recent works jointly with Bo-Yong Chen.

New sharp sampling theorems in (quasi) shift-invariant spaces

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Given a separated set Γ in $\mathbb{R}^d$ and a function g from $L^2(\mathbb{R}^d)$. The quasi shift-invariant space $V^p_\Gamma(g)$ generated by function g consists of all functions of the form

$$f(x) = \sum_{\gamma\in\Gamma} c_\gamma g(x-\gamma), \quad \{c_\gamma\} \in l^p(\Gamma).$$

This space is called shift-invariant if $\Gamma = \mathbb{Z}^d$.

Consider (quasi) shift-invariant spaces generated by

- bivariate Gaussian kernel:

$$g(x,y) = e^{-\alpha(x^2+y^2)}, \quad (x,y) \in \mathbb{R}^2;$$

- function from the following families of generators that admit meromorphic extension to $\mathbb{C}$:

$$g \in \mathcal{C}(\alpha) = \left\{ \frac{P(e^{\alpha x})}{Q(e^{\alpha x})} \right\} \quad \text{or} \quad g \in \mathcal{K}(\alpha) = \left\{ \frac{e^{-\alpha x^2/2} P(e^{\alpha x})}{Q(e^{\alpha x})} \right\}, \quad x \in \mathbb{R},$$

where $\alpha > 0$ and P, Q are polynomials satisfying some natural assumptions.

We will discuss new sharp sampling theorems for the quasi shift-invariant spaces $V^p_\Gamma(g), g \in \mathcal{C}(\alpha)$, and shift-invariant spaces $V^p_{\mathbb{Z}}(g)$ and $V^p_{\mathbb{Z}^2}(g)$ generated by $g \in \mathcal{K}(\alpha)$ and by a bivariate Gaussian kernel respectively. As an application, we obtain new results on the density of semi-regular lattices of the corresponding Gabor frames.

The talk is based on joint papers with A. Ulanovskii (University of Stavanger) and J.L. Romero (University of Vienna).

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