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# Optimal exploitation of a renewable resource with capital limitations.

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**Abstract**

A model of interaction between a renewable natural resource with capital limitations, as exemplified by the optimal investment problem of sheep farming in a Nordic context, is analyzed. The model builds on existing studies from the fisheries literature, but the important difference is that while capital is related to harvesting effort in the fisheries, capital attributes to production capacity to keep the animal stock during the winter in our farm model. The paper provides several results where both optimal steady states and the optimal approach paths are characterized analytically. The results are further supported by a numerical example.

**Keywords:**

Livestock management, Irreversible investment, Sheep farming, Optimal control, Singular solutions.

## 1.Introduction

Following the pioneering work of Smith (1968), economic models of renewable resource management have occasionally been extended to include investment in man-made capital. Even though most, if not all, contributions to this strand of literature have been related to fishery management problems, spurred by the seminal contribution of Clark et al. (1979), much of the conclusions obtained here can probably quite easily be extended into the management of other types of wild natural resources, like terrestrial wildlife. In this paper, we look at another type of renewable management problem with capital limitations, namely domestic livestock management. The important difference is that while capital determines the fishing effort in the fishery problem, capital is related to the capacity to keeping animals during the indoors winter season in our farm problem, which is exemplified by sheep farming in a Nordic context.

The literature on the management of what may be viewed as two capital stocks, one man-made and the other one biological, is quite small. Clark et al. (1979) emphasized the irreversibility of investment, meaning that man-made capital cannot be sold once having been bought, and they showed how the possible approach paths towards the optimal steady state is greatly affected by this property. Their model is linear in both controls, investment in fishing vessels and harvest of the fish stock, and the approach paths are therefore characterized by a combination of bang-bang and singular controls. Stochastic elements are included in a paper by Charles and Munro (1985), and McKelvey (1985) analyzes open access dynamics in a fishery with man-made capital. Boyce (1995) formulates a similar model to that of Clark et al. (1976), but with non-linear investment costs. He finds, not surprisingly, that the derived optimal approach path is no longer of the bang-bang type. Sandal et al. (2007) extend the literature with a model without any non-negative constraint on investment, but where capital is less valuable when sold than when bought.

In this paper we analyze the optimal investment and harvest, or stocking, decision problem of a sheep farmer. The farmer, assumed to be well-informed and rational, aims to maximize present-value profit generated by meat production. The market price of meat is taken as given, as we consider a single farm, and abstract from both exogenous price fluctuations and (other) stochastic factors such as climatic variations. In addition to the natural capital stock, the animals, the farmer must also hold a certain amount of man-made capital which adheres to the familiar mechanisms of investment and depreciation, to keep the animals indoors during the winter season. Man-made capital in this farming system is thus mainly buildings and related equipment which is instrumental in determining farm capacity. We are not aware of other theoretic domestic livestock management models that include man-made capital in addition to animal capital, even though capital theoretical treatments of livestock are frequently found within the resource economics literature, see e.g. Kennedy (1986). Farm models include Jarvis (1974) who formulated a timing problem of cattle grazing, and Skonhoft (2008) who analyzed the optimal stocking problem of Nordic sheep farming. Our model and reasoning builds to some extent on this last paper, but Skonhoft studied a situation with no man-made capital limitations and with different year classes of the animal capital. Different year classes are not included in the present paper. The research problem here is to find the optimal slaughtering and investment policy in such a Nordic farming system, and to characterize both the optimal steady state and approach paths.. In the subsequent analysis, natural and man-made capital will generally be referred to merely as ‘animals’ and ‘capital’, respectively.

The rest of the paper is organized as follows. Section 2 describes briefly the Nordic sheep farming system and the model is formulated. Section 3 describes the optimal solution to the model while we in section 4 proceed to analyze the steady states. Having identified the optimal steady states, we analyze in section 5 the approach path and show that it involves a combination of bang-bang and singular controls. Numerical simulations are shown in section 6, while section 7 concludes the paper.

## 2. Model

The following analysis is related to economic and ecological conditions found in Norway, but these also exist in Iceland and Greenland. There are about 2.1 million sheep in Norway during the summer grazing season, divided among some 16,000 family farms. The average farm size is therefore quite small and accounts only for about 130 animals. Sheep farms are located either close to mountain areas and other sparsely populated areas or along the coast, with a means to transport the animals to more distant alpine areas with access to areas of summer grazing land. Such land is typically communally owned and managed. There is a sharp distinction between the summer grazing season and the winter indoors season. While food is abundant during the summer grazing season, housing and indoor feeding is required throughout winter because of snow and harsh weather conditions. The indoors winter season is typically from mid-October to the beginning of May next year. The adult sheep and the newborn lambs are then released for outdoors grazing. In September- October slaughtering takes place. In Norway, winter feeding basically consists of hay grown on pastures close to farms, with the addition of concentrate pellets provided by the industry. The main product is meat, which accounts for about 80% of the average farmer's income. The remainder comes from wool, because sheep milk production is virtually nonexistent (Nersten et al. 2003). However, the income from wool is neglected in the following analysis.

We begin with formulating the animal growth equation, given in discrete time, and where additions to the stock occur once a year (in the spring), as does the harvest of animals (in the fall). As our main focus is on the interaction between biological and man-made capital, we do not distinguish between different age classes of animals (but see Skonhoft 2008), but consider a biomass model where ‘a sheep is a sheep’. The rate of growth in animal biomass is further assumed constant, as is reasonable with a domestic animal stock facing controlled breeding and maintenance; that is, there is no density dependent growth process. The growth function for animal biomass is thus given as:

(1) $X_{t+1} - X_t = rX_t - H_t,$

where $X_t$ is the animal stock size at time (year) $t$, $H_t$ is harvest and $r > 0$ is the animal stock growth rate, assumed to be constant. A feature of Scandinavian sheep farming is that live animal are generally not traded. Therefore, we do not consider the possibility of restocking and require $H_t \geq 0$.

Man-made capital, also assumed to be homogenous, is used as housing for the animal stock during the winter indoors season. Each year, a constant positive amount of investment is allowed, and a constant fraction of the capital stock depreciates due to wear and tear. The net capital growth is thus given by:

(2) $K_{t+1} - K_t = -\gamma K_t + I_t,$

where $K_t$ is the capital stock and $I_t$ is the accompanying (gross) investment. $\gamma > 0$ is the rate of depreciation, assumed to be fixed.

The revenue of the farmer is made up of income from meat production. With $p > 0$ as the slaughtering price (net of slaughtering costs), the current meat income for the farmer simply reads $pH_t$ and is included as the first term in the profit equation:

(3) $\Pi(H_t, X_t, K_t, I_t) = pH_t - V(X_t) - Q(X_t, K_t) - cI_t$.

$p$ is assumed fixed over time and independent of the harvest decision, as explained above (section 1).

We then have the cost side, where we first describe at the operating cost. The operating cost structure differs sharply between the outdoor grazing season and the indoor feeding season. As explained, during the grazing period the sheep may graze on communally owned lands ('commons') or private land. Within the Nordic sheep farming system, such land may be available cost free, or the farmer may pay a fixed annual rent (Austrheim et al. 2008). The variable cost is hence simply assumed to be the indoor season operating cost. These costs, which include labor cost (typically as an opportunity cost), electricity and veterinary costs in addition to fodder, are assumed to determined uniquely by the size of the animal stock, i.e., $V_t = V(X_t)$, and with $V' > 0$, $V'' > 0$, and $V(0) = 0$. The argument for a strictly convex cost function is that fodder production is constrained by the size of the available land; that is, as the stock becomes larger it becomes progressively more costly to provide fodder.

As mentioned, in contrast to what is found in the fisheries literature where capital normally is equivalent to harvesting effort (e.g., Clark et al. 1979), capital in our farm system is housing and related equipment to keep the animal stock during the winter. We assume that there is no absolute constraint on the amount of animals that a given amount of capital can support, so that there is no such thing as 'full' capacity utilization in our farm model. However, as the indoor space per animal diminishes, the operating procedure becomes increasingly cumbersome. We hence include the capacity utilization cost, or congestion cost, function $Q(X_t, K_t)$ in our current profit equation (3). It increases with the number of animals, for any given amount of capital, such that $Q_x > 0$ and $Q_{xx} > 0$, together with $Q_K < 0$, $Q_{KK} > 0$ and $Q_{KX} < 0$. In addition, we have $Q(0, K_t) = 0$ when $K_t > 0$ and $\lim_{K_t \to 0} Q(X_t, K_t) = \infty$ for $X_t > 0$. For all positive stock values this function is hence convex in $X_t$ and $K_t$. In the numerical section 6 below we specify this cost function.

The final cost component is the cost of buying new capital equipment. We assume that there is a constraint on the size of investment in each period, due to, say, limited access to credit, such that $I_t \leq I^{\max}$. The cost per unit of investment is fixed and given by $c > 0$, so that the yearly investment cost reads $cI_t$. An alternative assumption, following e.g. Sandal et al. (2007) , could have been to introduce adjustment costs to limit the amount of investment carried out in each time period. In our model, as in reality, investment is also irreversible; the buildings cannot be sold once having been set up; that is, $I_t \geq 0$.

**3. Optimal management**

The farmer aims to maximize present-value profit subject to the dynamic constraints imposed by the growth equations for animals (1) and capital (2), and the constraints on harvest and investment in each period. We suppose an infinite planning horizon, meaning that we are looking for an optimal steady state. The planning problem of the farmer is then formulated as:

$$
(4) \qquad
\begin{aligned}
& max\left\{\sum_{t=0}^{\infty}\rho^{t}\left[pH_{t}-V\left(X_{t}\right)-Q\left(X_{t},K_{t}\right)-cI_{t}\right]\right\} \\
& s.t. \quad X_{t+1}-X_{t}=rX_{t}-H_{t} \\
& \qquad\; K_{t+1}-K_{t}=-\gamma K_{t}+I_{t} \\
& \qquad\; 0\leq H_{t},\ 0\leq I_{t}\leq I^{max} \\
& \qquad\; X_{t},K_{t}>0 \\
& \qquad\; X_{0},K_{0}\ \text{given}
\end{aligned}
,
$$

and where $\rho = 1/(1+\delta)$ is the discount factor with $\delta \geq 0$ as the constant discount rate.

The Lagrangean of this problem may be written as:

$$
\begin{aligned}
L=\sum_{t=0}^{\infty}\rho^{t}\ \Big\{ & pH_{t}-cI_{t}-V\left(X_{t}\right)-Q\left(X_{t},K_{t}\right) \\
& -\rho\lambda_{t+1}\left[X_{t+1}-\left(1+r\right)X_{t}+H_{t}\right] \\
& -\rho\mu_{t+1}\left[K_{t+1}-\left(1-\gamma\right)K_{t}-I_{t}\ \right]\ \Big\},
\end{aligned}
$$

where $\lambda_t$ and $\mu_t$ are the shadow prices of the animal and capital stock, respectively. The necessary conditions for a maximum are:

$$(5) \qquad \frac{\partial L}{\partial H_{t}}=p-\rho\lambda_{t+1}\leq 0,\ \ 0\leq H_{t}$$

$$(6) \qquad \frac{\partial L}{\partial I_{t}}=-c+\rho\mu_{t+1}\gtreqless 0,\ \ 0\leq I_{t}\leq I^{\max}$$

$$(7) \qquad \frac{\partial L}{\partial X_{t}}=-V'-Q_{X}+\rho\lambda_{t+1}\left(1+r\right)-\lambda_{t}=0$$

$$(8) \qquad \frac{\partial L}{\partial K_{t}}=-Q_{K}+\rho\mu_{t+1}\left(1-\gamma\right)-\mu_{t}=0$$

These conditions are also sufficient if the Lagrangean is concave in the states and controls jointly. Since the Lagrangean is linear in the controls, the sufficiency conditions boil down to $L_{XX}=-\left(V''+Q_{XX}\right)\leq 0,\ \ L_{KK}=-Q_{KK}\leq 0$ and $L_{XX}L_{QQ}-L_{XY}{}^{2}=Q_{KK}\left(V''+Q_{XX}\right)-Q_{KX}{}^{2}\geq 0$, which are satisfied for the given properties of the cost functions. The transversality conditions for the infinite horizon problem must also hold; i.e., $\lim_{t \to \infty} \lambda_t \left( X_t - X^* \right) \geq 0$ and $\lim_{t \to \infty} \mu_t \left( K_t - K^* \right) \geq 0$ and where the asterisk indicates optimal steady state values.

The interpretation of (5) is that harvest is set to zero whenever the price of meat is lower than the discounted shadow price of the animal stock, and positive otherwise. Similarly, (6) states that there will be positive investment only when the unit investment cost is lower than the discounted shadow price of capital. It can be set to its maximum value, or it can be set to the interior of the control region. This 'singular' control policy is only implicitly defined from the first order conditions. We thus have the following alternatives for harvest policy, letting $H^S$ denote singular harvest:

$$H_t = \begin{cases} \infty & when \quad p > \rho \lambda_{t+1} \quad (impulse\ harvest) \\ H^S & when \quad p = \rho \lambda_{t+1} \\ 0 & when \quad p < \rho \lambda_{t+1} \end{cases}$$

Note that, since there is no upper bound on harvest except from the size of the stock itself, whenever $p > \rho \lambda_{t+1}$, the herd will be reduced immediately (that is, within one time period), to the level where $p = \rho \lambda_{t+1}$, and singular harvest takes over. This is often called impulse control in the optimal control theory literature.

As for investment, we get:

$$I_t = \begin{cases} I^{max} & when \quad c < \rho \mu_{t+1} \\ I^S & when \quad c = \rho \mu_{t+1} \\ 0 & when \quad c > \rho \mu_{t+1} \end{cases}$$

Therefore, investment will be at its maximum or minimum level whenever the per unit investment cost is lower or higher than the shadow price of capital. When the shadow price reaches the point where it equals the present value of the unit cost of investment, the control will either switch between the two control boundaries – going from maximum to zero investment, or vice versa - or stay at singular investment for some amount of time.

Singular control relationships for both stocks can be further derived from the first order conditions. If singular harvest holds, we have from equation (5) $p = \rho\lambda_{t+1}$, which means that the shadow price of the animal stock is constant, and equalizes $\lambda^* = p / \rho$. When inserted into condition (7), we find the following golden rule condition for the animal stock:

(9) $$\left(r-\delta\right)p = V'(X) + Q_X(X,K)$$

Equation (9) therefore describes the relationship between $X$ and $K$ that is consistent with singular harvest. This condition may also be written as $p = (1/\delta)\left[pr - V'(X) - Q_X(X,K)\right]$ indicating that the market revenue from selling one animal should equalize the discounted net benefit from keeping it. Because both $V'$ and $Q_X$ are positive, we must require that the animal growth rate exceeds the discount rate, $r > \delta$, which is a well-known condition for a positive steady state animal stock (see, e.g., Clark 1990). As both $r$ and $\delta$ are constant this must always hold, also outside the steady state.

With singular investment, we have from (6) $c = \rho\mu_{t+1}$, which means that the capital shadow price is constant, $\mu^* = c / \rho$. Inserted into equation (8) gives the golden rule condition for capital:

(10) $$\left(\gamma+\delta\right)c = -Q_K(X,K).$$

Equation (10) defines a relationship between $X$ and $K$ that is consistent with singular investment. It may also be written as $c = (1/\delta)[-Q_K(X,K) - \gamma]$ indicating that the unit investment cost should equalize the discounted marginal net benefit from holding capital.

## 4. The steady states

In an interior equilibrium where both $H^* = H^S\left(K^*, X^*\right)$ and $I^* = I^S\left(K^*, X^*\right)$, the golden rule conditions (9) as well as (10) must hold. But in principle, one, or both, controls may also be set at a boundary at a steady state. From equation (1), however, as long as the rate of animal growth is positive and constant, the harvest rate must be positive in the steady state. Since there is no upper constraint on harvest, the steady state harvest policy must then be singular. From equation (2), steady state investment must also be positive, but may be set to its maximum level, where the gross investment in each year equals depreciation, keeping the capital stock at its optimal steady state level given the investment constraint. We therefore have two alternatives for the steady state, and this is stated as:

**Result 1**. There are two steady state alternatives. The first is interior where both controls are singular. In the second harvest is singular while investment is at the maximum level.

We first study the interior steady state in some detail and then discuss the situation where the investment constraint binds. At an interior steady state, the two schedules defined by equations (9) and (10) must intersect. Except from the very special case where the two curves are coinciding, there can be at most a countable number of equilibria. When differentiating (9) and (10) we find $dX / dK = -Q_{XK} / (V'' + Q_{XX}) > 0$ and $dX / dK = -Q_{KK} / Q_{XK} > 0$, respectively. Therefore, both schedules (9) and (10) slope upwards in the $\left(K, X\right)$-space, but the curvatures cannot be determined generally without imposing restrictions on third derivatives. This allows for an arbitrary number of intersection points, with a correspondingly arbitrary number of stable and unstable equilibria. However, we find a stable steady state $K^*$ and $X^*$ where the optimal harvest condition (9) intersects the singular investment schedule (10) from above, so that $-Q_{XK} / (V"+Q_{XX}) > -Q_{KK} / Q_{XK}$ holds at the intersection point. Otherwise, the intersection point is an unstable equilibrium. This holds because a local maximum is found where the Lagrangean is concave around a stationary point which requires that the condition $\left(V"+Q_{XX}\right)Q_{KK} - Q_{KX}{}^2 = \Gamma(X,K) \geq 0$ must be satisfied. Rearranging this expression gives $-Q_{XK} / (V_{XX} + Q_{XX}) > -Q_{KK} / Q_{XK}$ as claimed. When these equilibria are found, the steady state harvest follows from (1) as:

(11) $$H^* = rX^*,$$

and the steady state investment from (2) as:

(12) $$I^* = \gamma K^*.$$

Equations (9) and (10) and the sufficiency condition can be used to derive some comparative static results about $X^*$ and $K^*$. In a next step, the effects on $H^*$ and $I^*$ follow recursively from equations (11) and (12), respectively. We first look at the effect of a changing meat price and when differentiating (9) and (10) we find $\left(r-\delta\right)dp = \left(V'' + Q_{XX}\right)dX + Q_{KX}dK$ and $0 = -Q_{KK}dK - Q_{KX}dX$, respectively. Combing these expressions yields the partial price effects $\partial X^* / \partial p = Q_{KK}\left(r-\delta\right) / \Gamma\left(X,K\right) > 0$ and $\partial K^* / \partial p = -Q_{KX}\left(r-\delta\right) / \Gamma\left(X,K\right) > 0$. This is stated as:

**Result 2**. An increase in the price of meat will result in a larger stock of animals and man-made capital in an interior optimal steady state.

This result is the opposite of what is found in the standard fishery model (e.g., Clark 1990) where a price increase leads to more aggressive harvest and a lower optimal steady state stock. The reason for the opposite result in our farm model is that costs here are not associated with harvest, but with stock maintenance. With a higher meat price, the farmer thus finds it beneficial to keep a higher stock of both types of capital as the relative maintenance costs decreases. Differentiating (9) and (10) also gives information about the effects of a change in the discount rate. We find positive effects for both stocks, and this is stated as:

**Result 3.** An increase in the discount rate leads to reduced stocks of both animals and capital in the interior optimal steady state.

This result fits conventional economic intuition, but is far from obvious when more than one capital stock is considered and cannot be deduced form the golden rule conditions directly. As shown by e.g. Asheim (2008), paradoxical effects of discounting, such as a positive relationship between discounting and steady state consumption, may result from multi-dimensional models. Also, when there is a trade-off between the two stocks across alternative steady states, something that is typical for predator-prey models, one of the stocks must increase with discounting while the other goes down. However, this does not happen here.

Following the same procedure with respect to the other parameters, all the time assuming that the sufficiency conditions are fulfilled, the other comparative static results can also be computed. All results are reported in Table I where the investment and harvest effects are included as well. An increase in the investments cost or depreciation rate means that it is beneficial for the farmer to reduce the steady state animal and man-made capital stocks, whereas an increase in the growth rate of animals leads to larger optimal stocks of both animals and capital. These results are more or less as expected, and the effects of the parameters work in the same direction for both stocks. It can also be confirmed that we find the combined effects $\partial X^* / \partial r = \partial X^* / d\gamma - \partial X^* / \partial \delta > 0$, indicating that the negative effect on the animal stock of higher depreciation rate must be smaller than the negative effect of higher discount rate. We also find $\partial K^* / \partial r = \partial K^* / d\gamma - \partial K^* / \partial \delta > 0$, indicating the same different effects on the optimal steady state capital stock.

The effects on the control variables through equations (11) and (12) are quite straightforward, except for the animal growth rate and rate of depreciation which both have direct and indirect effects on the steady state harvest and investment, respectively. For a positive shift in the animal growth rate, the two effects work in the same direction and lead to higher harvest in the steady state, since from (11) $\partial H^* / \partial r = X^* + r \partial X^* / \partial r > 0$. With the depreciation rate, however, the two effects work in the opposite direction as we find $\partial I^* / \partial \gamma = K^* + \gamma \partial K^* / \partial \gamma$ from equation (12). The direct effect is to increase the required amount of investment to maintain a given amount of capital, whereas the indirect effect is to decrease the optimal steady state capital stock. The overall effect is ambiguous with general functional forms.

[Table I about here]

We then consider the other steady state possibility where the investment is no longer singular, and hence the condition

(13) $I^* = I^{\max} < I^S$

replaces equation (12) in the interior steady state solution. The steady state capital stock now follows directly through (12) as $K^* = I^{max} / \gamma$ , which inserted into equation (9) yields the steady state animal stock. The amount of capital will now for obvious reasons be below what was found in the interior steady state. Because condition (9) yields a positive relationship between the two stocks, the number of animals will also be below what was found in the interior steady state. The steady state harvest again follows from equation (11) as $H^* = rX^*$, and the number of animals slaughtered will consequently also be below the previous steady state.

The comparative static results are now somewhat different, as indicated with brackets in Table I. When the investment constraint binds, the only factor that affects the steady state capital stock is the depreciation rate. The sign of the effects on the animal stock are as before, except for the investment cost which now has zero effect.

## 5. Optimal approach paths

We have characterized the two alternatives for an optimal steady state, both the interior solution and the case where the upper investment constraint binds at the optimum. The next task is to study the optimal approach paths. In general, approach paths in multi-dimensional models are often complicated to analyze, as exemplified by the predator-prey model of Mesterton-Gibbons (1996). For a more recent example, see Horan (2005). We find, however, that in our case it is possible to derive an intuitive solution which is easily explained graphically. As indicated above, the optimal trajectories result from a combination of extreme and singular controls. We know that both controls can be singular simultaneously only at an interior steady state, so that one of the control constraints must always bind outside an equilibrium. Whenever the animal stock is above the $H^S$-schedule, it will be harvested down instantaneously (or more precisely, within on time period), until the $H^S$-schedule is reached since $H_t$ is unconstrained from above. Then either i) the system will follow the $H^S$-schedule, with singular harvest for a period of time, or ii) harvest is set to zero, in which case the $H^S$-schedule acts as a switch between extreme controls. Ignoring the case where $H_t > H^S$, which is impossible for any more than one time period, we now consider the various alternative control regimes. The different cases can be best understood with reference to Figure 1 where a situation with a unique interior equilibrium is depicted. The singular control schedules are in accordance with the specific functional forms used in numerical section 6 (Eqs. 9' and 10' below). Four different initial states, labeled A, B, C and D, are shown along with the optimal approach paths originating from them. Notice that initial states A and C have the same value for the animal stock, whereas initial states B and D represent the same capital stock value. These properties are further exploited in the numerical section. Panel a) demonstrates the first three cases, where the upper investment constraint does not bind along the approach path.

Case 1: $H = 0$, $I = I^S$. The only possibility when investment is singular outside of the steady state is that harvest is zero. This happens when, as from an initial situation such as A or B, the initial capital stock is below the steady state level and the system has been controlled to reach the singular investment schedule. The system will then follow the $I^S$-schedule (10) towards the steady state, when the investment constraint does not bind.

Case 2: $H = H^S$, $I = 0$. From an initial situation such as point D where both stocks are above their interior steady state levels, the animal stock is harvested down until the $H^S$-schedule (9) is reached, and the system moves leftwards along the $H^S$-schedule towards the equilibrium. Note that the $I^S$-schedule plays no role here, and is therefore represented by a dashed line in the figure.

Case 3: $H = 0$, $I = 0$. Here both controls are set to zero, which happens when the state of the system is below both singular control schedules, as at point C. This control regime continues until one of the two singular control schedules is reached, and one of the two above alternatives takes over.

The next two 'intermediate' cases, where the upper investment constraint prevents the system from following the $I^S$-schedule, are shown in panel b) Figure 1. Both these cases depend on the system having reached either the $H^S$-schedule or the $I^S$-schedule below and to the left of the equilibrium. This may happen if the initial states are given by points A or B.

Case 4: $H = 0$, $I = I^{\max}$. When $H^S < 0$, meaning that following the $H^S$-schedule would require restocking of animals, which is omitted in our model, and $I^S > I^{max}$, so that the maximum investment constraint does not allow the system to follow the $I^S$-schedule either, the state of the system will be somewhere below the $H^S$-schedule and above the $I^S$-schedule.

Case 5: $H = H^S$, $I = I^{max}$. This situation arises when the system has reached the $H^S$-schedule (9), either after an initial impulse harvest, or from a situation such as in case 4, but the maximum per period investment is not sufficiently large to detract the system from the $H^S$-schedule. The system will then follow the $H^S$-schedule to the steady state.

The last situation to consider is the alternative steady state where the upper investment constraint is binding. Panel c) Figure 1 demonstrates. The equilibrium can now be found as a point on the $H^S$- schedule, below and to the left of the intersection point, with investment set to its maximum level at every point in time, $I^* = I^{\max} < I^S$. As shown in the figure, the approach path is along the $H^S$-schedule from both directions in this case. However, for sufficiently low stock values it is still possible to follow the $I^S$-schedule, as the animal stock growth is a constant share of the animal stock size, whereas the maximum investment is assumed to be independent of the size of the existing capital stock.

The different control scenarios can be further characterized by dividing the state-state space into three regions, with three different transitional control regimes. Region I: Above the $H^S$-schedule; impulse harvest, Region II: Below the $H^S$-schedule, above the $I^S$-schedule; $H = 0$, $I = I^{max}$ and Region III: Below both schedules; $H = I = 0$.

As is evident from Figure 1, the singular control schedules act either as switch lines or approach paths, depending on the control constraints. The approach path is identified as a bold line, which in panel a) consists of the part of the $I^S$-schedule (10) that is to the left of the optimal steady state, and the part of the $H^S$-schedule (9) that is to the right of the equilibrium. The upper constraint on investment may also entail that the $H^S$-schedule must be followed even from the left, at least when the equilibrium is sufficiently close. This situation is depicted in panels b) and c) Figure 1.

[Figure 1 about here]

Whenever the initial point is above the singular harvest schedule, a situation exemplified by points A and D in Region I, the stock will be slaughtered down immediately until the $H^s$-schedule is reached. If the state of the system is now above the singular investment schedule, the $H^S$-schedule acts as a switch and harvest is set to zero, as is the case when starting from point A. If not, the rest of the approach path is along the $H^S$-schedule, as with the trajectory from point D. When starting from below both schedules, as from points B and C, the singular approach path is the one of the control schedules that is encountered first, after a period with zero harvest and investment. As indicated in Figure 1a), the approach path is thus the lower one of the two singular control schedules, when feasible. As seen on Figures 1b) and 1c), however, a part of this control path may not be feasible if the upper investment constraint binds. In this case only the leftmost part of the $I^S$-schedule can be followed, while the equilibrium is encountered along the $H^S$-schedule from both directions. The $I^S$-schedule may then act partly as a switch between zero and maximum investment, which the trajectory from point B indicates.

By taking the first order difference of eqs. (9) and (10) , and using the growth equations to substitute for $K_{t+1} - K_t$ and $X_{t+1} - X_t$, we can also derive explicit feedback rules for both stocks. Recalling that $H_t = 0$ along the singular investment schedule, singular investment is given by:

$$I^S = \gamma K_t + \frac{V'' + Q_{XK}}{-Q_{KK}} rX_t,$$

Where the coefficient for $X$ is positive. Singular investment therefore depends positively on depreciation, and also on animal stock growth. A higher stock of animals, and/or a higher animal growth rate means that investment must increase to let capital growth keep pace with growth in the animal stock. Singular harvest is given by:

$$H^S = \begin{cases} rX_t + \dfrac{-Q_{KK}}{Q_{KX}} \gamma K_t & X_t > X^* \\ rX_t + \dfrac{-Q_{KK}}{Q_{KX}} \left( \gamma K_t - I^{\max} \right) & X_t < X^* \end{cases},$$

where the coefficient for $K_t$ is positive, and with a similar interpretation. Note that the singular harvest rule is different depending on whether the system is on the $H^S$-schedule to the left or right of the equilibrium, as investment will be set to $I^{\max}$ or zero in the two situations, respectively.

The next two results regard the monotonicity of the approach paths, and are related to results from the fisheries literature. First observe that the approach path is monotonous with respect to both $X$ and $K$ along the singular control schedules and in Region II (see the discussion above). Also note that a) In Region I there is only impulse harvest and no investment, and b) in Region III the monotonous part of approach path may be encountered on either side of the equilibrium if $K_0 > K^*$ and $X_0 < X^*$. The first of these results is stated as:

**Result 4.** It is never optimal for capital to overshoot its optimal steady state level. However, capital may undershoot the steady state if $X_0 < X^*$.

Proof: Positive investment cannot occur in any of the two regions outside the monotonous part of the approach path. Hence, overshooting is impossible. Undershooting happens if, from Region III with $K_0 > K^*$ and $X_0 < X^*$, the monotonous part of the approach path is reached where $K_t < K^*$.

**Corollary:** It will never be optimal to have excess capacity in the steady state.

This result differs from what is found by Clark et al. (1979), where it is optimal to have excess capacity in the steady state if the depreciation rate is zero. The reason that this does not happen here is that capital plays no role in the harvesting process. Therefore, it is not profitable, or possible, to speed up the approach to the equilibrium by overinvesting, if the initial animal stock is above the equilibrium level. The next result concerns the development of the animal stock:

**Result 5.** The animal stock may either undershoot or overshoot the optimum, depending on the initial situation.

Proof: When $X_0 > X^*$ and $K_0 < K^*$ in Region I, the animal stock will be reduced immediately until the monotonous part of the approach path is reached where $X_t < X^*$, implying undershooting. From Region III with $K_0 > K^*$ and $X_0 < X^*$, overshooting occurs if the monotonous part of the approach path is reached where $X_t > X^*$.

This also contrasts the Clark et al. (1979) model, and subsequent contributions within the fisheries literature. The intuition is that a more profitable rate of capacity utilization can be obtained by temporarily reducing the animal stock below the steady state level if the capital stock is low, and expanding it beyond the steady state level if the capital stock is large. Both situations depend on the fact that the capital stock cannot be adjusted instantaneously in either direction. The next result is stated without proof as:

**Result 6**: If the upper investment constraint is not binding on the approach path, the optimal steady state will almost always be approached with one control set at the interior and the other at zero.

In principle, all control combinations are possible approaches to the equilibrium, but the case where $X_0 > X^*$ and $K_0 = K^*$, so that the equilibrium is reached by a one-time slaughtering down of the animal stock only, and the case where the equilibrium is reached by setting both controls to zero, can both only be satisfied by a fluke. The general approach is along one of the singular control schedules, and ruling out the possibility that $I = I^{\max}$ along the approach path, the non-singular of the controls must be zero.

## 6. Numerical example

To shed some further light on the above analysis, the model is now illustrated numerically[1]. We do not attempt to accurately describe the economic situation of a Nordic sheep farmer, but to demonstrate the workings of the model with reasonably realistic parameter values. First, we specify the functional forms. The congestion cost function is specified as:

(11) $$Q\left(X_t, K_t\right) = \frac{\theta}{2K_t} X_t^2,$$

where $\theta > 0$. It is readily confirmed that this cost function satisfies the properties stated in section 2 above. The operating cost function is next specified as:

(12) $$V(X_t) = \frac{\eta}{2} X_t^2,$$

with $\eta > 0$. With these functional forms, we find the following expression for the singular harvest and investment schedules:

(9') $$\left(r - \delta\right) p = \left(\eta + \frac{\theta}{K_t}\right) X_t,$$

and

(10') $$\left(\gamma + \delta\right) c = \frac{\theta}{2}\left(\frac{X_t}{K_t}\right)^2.$$

It is easily recognized that both schedules start form the origin and have a positive slope. While the singular investment schedule (Eq. 10') is a straight line, the singular harvest schedule (Eq. 9') yields $X$ as a strictly concave function of $K$, cf. Figure 1 above. They have

[1] The numerical optimization was performed using the KNITRO for MATLAB solver form Ziena Optimization, with MATLAB release 2011b.

thus one interior intersection point, provided that the $H^S$-schedule is steeper than the $I^S$-schedule at the origin, which corresponds to a stable equilibrium (see also section 4 above).

The numerical optimization is performed using the parameter values found in Table II. As indicated, the discount rate, the growth rate for the animals and the indoor feeding cost parameter are taken from Skonhoft (2008) while the depreciation rate is what is used by Statistics Norway for buildings (Statistics Norway 2011). The investment and congestion cost parameters are calibrated for our model such that the number of animals in the steady state should represent a medium sized Norwegian farm. In addition, we assume that the maximum yearly investment is fixed at $I^{\max} = 20$ (m$^2$).

[Table II about here]

Table III demonstrates the steady state results. The steady state is interior, as the depreciation is below the investment constraint; $\gamma K^* = 8 < I^{\max}$, and can be found as the solution to equations (9') and (10'). The results with the baseline parameter values are shown in the first column, while the next column shows the results of a 50% increase in the meat price, to 360 (EUR/animal), while all the other parameters are kept at their baseline values. In the last column the discount rate is increased by 50%, to $\delta = 0.06$. In the baseline calculation, the optimal stocking is 120 (animals), the capital stock becomes 200 (m$^2$) while the profit is about 9,800 (EUR). The change in the discount rate has a modest impact on the optimal steady state animal stock level while the effect on the capital stock is somewhat more substantial (see also the comparative static results Table I). The profit is only modestly affected. The slaughter price change, on the other hand, strongly affects the profit which is more than doubled compared to the baseline alternative. The optimal animal stock level is increased by 60%.

Note also that capacity utilization is unaffected when the price shifts up. The reason is that the $I^S$-schedule (10') is linear and not affected by the slaughter price. A change in the discount rate, on the other hand, shifts this schedule as well as schedule (9') and hence the rate of capacity utilization changes.

[Table III about here]

The dynamics are demonstrated in Figures 2 and 3 where the panels to the left are for the state variables while the panels to the right depict the corresponding harvest and investment paths. Four different initial situations are considered that correspond roughly to the initial states depicted in Figure 1, such that $(K_0, X_0)$ assumes the values $(50,110)$ at point A, $(210,20)$ at point B, $(300,110)$ at point C and $(210,200)$ at point D.

In Figure 2, the approach paths from points A and C are depicted. In both these initial situations the animal stock size is 110 (animals), which is close to the optimum (Table III). The initial capital stock is either far below (A), or well above (C) its steady state level. This figure illustrates the possibilities for the animal stock to over- or undershoot the steady state. If $K_0 = 50$, an immediate harvesting down of the animal stock is followed by a combination of maximum investment and singular harvest until the equilibrium is reached after about 12 years. Since the investment constraint is binding along the approach path, but not in the steady state, the different trajectories corresponds to the ones depicted in panel b) in Figure 1. From $K_0 = 300$, both controls are set to zero and the animal stock grows past its steady state level before the singular harvest schedule is followed, with zero investment.

[Figure 2 about here]

Figure 3 demonstrates the optimal paths when the initial states are found as points B and D. With $K_0 = 210$ in both cases, the initial capital stock is close to the optimum, while the animal stock level is either far below (initial situation B with $X_0 = 20$), or far above (D with $X_0 = 200$), the optimal steady state. In either case, the steady state is reached faster than in Figure 2, as both growth and reduction in the animal stock is faster than for the capital stock. Initial point B entails zero harvest and investment, followed by one period of maximum investment (Region II in Figure 1b) before the equilibrium is encountered. From initial situation D, impulse harvest and depreciation of the capital stock leads to the equilibrium after just two time periods. From point B, the capital stock undershoots the steady state (but may never overshoot, as discussed in section 5).

[Figure 3 about here]

**7. Concluding remarks**

In this paper we have from a theoretical point of view, analyzed the dynamic optimization problem of a profit maximizing farmer who possesses both animals and man-made capital. The model builds on existing studies from the fisheries literature, but the important difference is that while capital is related to harvesting effort in the fisheries, capital attributes to production capacity to keep the animal stock during the winter in our farm model. The linearity of the model allows an intuitive graphical description that is rare in multi-dimensional optimization problems. Both the steady state and the optimal approach paths have been characterized analytically, and demonstrated by a numerical example related to Scandinavian sheep farming.

The steady state was shown to be either an interior optimum with interior controls, or a constrained optimum with investment set to its maximum value (Result 1). The effects of parameter changes were studied analytically. We found that with a higher meat price the farmer will find it beneficial to increase the stock of animals as well as the amount of capital in the interior steady state (Result 2), while an increase in the discount rate yields opposite effects (Result 3).

As the objective function is linear in both control variables, the approach path is a combination of bang-bang and singular controls, and along the approach path at most one of the controls is singular. The dynamics are different from what is found in the typical fishery models, as in particular there will be a gradual building up of capital, not a one-time impulse investment where the capital stock overshoots the steady state. With capital, only undershooting is possible (Result 4). The animal stock may, on the other hand, both over- and undershoot the optimal steady state (Result 5). In general, one of the controls will be singular along the approach path while the other is set to zero, if the upper investment constraint does not bind (Result 6).

We have focused on a situation with a unique interior equilibrium. However, with different specifications of the cost function there may be several equlibria. With a positive discount rate, the choice of steady state will then in general depend on the initial situation, so that the system is history dependent. The dynamics of such a system will be a rather straightforward generalization of the system analyzed here however, once the optimal steady state is identified. Another possible extension is to include an absolute limit on the number of animals per square meter of housing, typically set by authorities to secure animal health. If this constraint binds along the approach path it will imply maximum investment together with positive harvest of animals. If the capacity utilization constraint is binding in the steady state, it will imply singular steady state harvest along with maximum investment.

The main contribution of this paper is related to the role of capital which is used here for maintaining the animals and hence plays no role in the harvesting process. In addition, we assume a domestic animal stock where the unit harvest cost is stock independent, and natural growth is density independent and hence also unaffected by stock size. Given that these assumptions also are valid in other types of production involving domestic renewable resources, the model here may have wider applications. Possible examples include other types of livestock management and other areas of modern agricultural production, as well as aquaculture.

## References

Asheim, G. (2008): Paradoxical consumption behavior when economic activity has environmental effects. Journal of Economic Behavior & Organization 65: 529-546.

Austrheim, G., L.J. Asheim, G. Bjarnason, J. Feilberg, A.M. Fosaa, Ø. Holand, K. Høegh, I.S. Jónsdóttir, B. Magnússon, L.E. Mortensen, A. Mysterud, E. Olsen, A. Skonhoft, G. Steinheim and A.G. Thórhallsdóttir (2008): Sheep grazing in the North Atlantic region: A long term perspective on management, resource economy and ecology. Report Zoology Ser. 2008, 3 Vitenskapsmuseet NTNU.

Boyce, J. R. (1995): Optimal capital accumulation in a fishery: a nonlinear irreversible investment model. Journal of Environmental Economics and Management 28: 324-39.

Kennedy (1986): Dynamic Programming. Applications to Agriculture and Natural Resources. Elsevier Science, London.

Charles, A. T. and G. R. Munro (1985): Irreversible investments and optimal fisheries management: A stocastic analysis. Marine Resource Economics 1: 247-64.

Clark, C. (1990): Mathematical Bioeconomics. Wiley Interscience, New York.

Clark, C., F. Clarke and Munro (1979): The optimal exploitation of renewable resource stocks: Problems of irreversible investments. Econometrica 47: 25-47.

Horan, R. and C. A. Wolf (2005): The economics of managing infectious wildlife disease. American Journal of Agricultural Economics 87: 537-551.

Jarvis, L. (1974): Cattle as capital goods and ranchers as portfolio managers: An application to the argentine cattle sector. Journal of Political Economics 82: 489-520.

McKelvey, R. (1985): Decentralized regulation of a common property renewable resource industry with irreversible investment. Journal of Environmental Economics and Management 12: 287-307.

Mesterton-Gibbons, M. (1996): A technique for finding optimal two-species harvesting policies. Ecological modeling 9: 235-244.

Nersten, N., A. Hegrenes, O. Skjelmo, and K. Stokke (2003): Saueholdet i Norge (Sheep Farming in Norway). Report, Norwegian Agricultural Economic Research, Oslo.

Sandal, L. K., S. I. Steinshamn, A. Hoff (2007): Irreversible investments revisited. Marine Resource Economics 22: 255-266.

Skonhoft, A. (2008): Sheep as capital and farmers as portfolio managers: A bioeconomic model of Scandinavian sheep farming. Agricultural Economics 38: 193-200

Smith, V. L. (1968): Economics of production from natural resources. American Economic Review 58: 409-31.

Statistics Norway (2011): Statistical yearbook of Norway. Statistics Norway, Oslo

**Tables and figures**

**Table I: Comparative static results interior steady state with singular harvest and investment. In parentheses the constrained steady state with $I^* = I^{\max}$.**

| | $p$ | $c$ | $\delta$ | $r$ | $\gamma$ |
|---|---|---|---|---|---|
| $X^*$ | $+(+)$ | $-(0)$ | $-(-)$ | $+(+)$ | $-(-)$ |
| $K^*$ | $+(0)$ | $-(0)$ | $-(0)$ | $+(0)$ | $-(-)$ |
| $H^*$ | $+(+)$ | $-(0)$ | $-(-)$ | $+(+)$ | $-(-)$ |
| $I^*$ | $+(0)$ | $-(0)$ | $-(0)$ | $+(0)$ | $\pm(0)$ |

**Table II: Baseline parameter values**

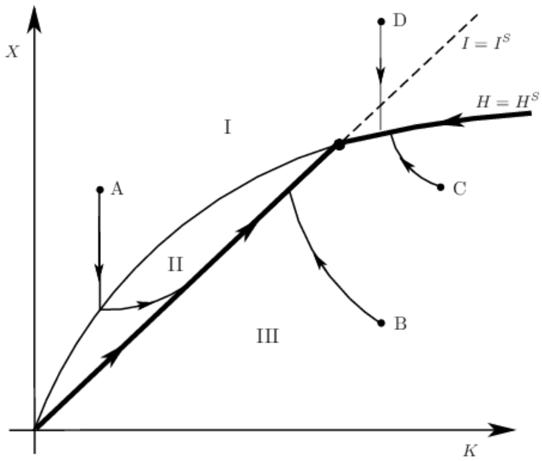
| Parameter | Description | Value |
|---|---|---|
| $\delta$ | Discount rate | 0.04 |
| $r$ | Animal growth rate | 0.7 |
| $c$ | Unit investment cost (EUR/m$^{2)}$ | 100 |
| $\eta$ | Feeding cost (EUR/animal$^2$) | 1.1 |
| $\theta$ | Congestion cost (EUR/(animal$^2$/m$^2$)) | 45 |
| $\gamma$ | Depreciation rate | 0.04 |
| $p$ | Meat price (EUR/animal) | 240 |

Sources: $r$, $\eta$ and $p$ based on Skonhoft (2008), $\gamma$ from Statistics Norway (2011), $c$ and $\theta$ calibrated.

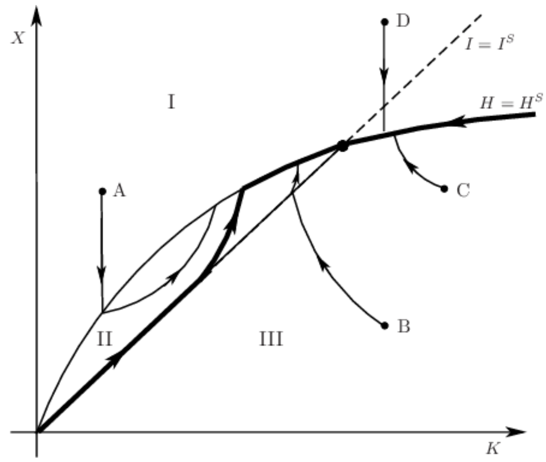
**Table III: Steady state results**

| | | Baseline | $p$ up 50% | $\delta$ up 50% |
|---|---|---|---|---|
| Animals | $X^*$ | 120 | 192 | 112 |
| Capital (m$^2$) | $K^*$ | 200 | 322 | 168 |
| Capacity utilization (animals/m$^2$) | $X^*/K^*$ | 0.60 | 0.60 | 0.67 |
| Slaughter income (EUR) | $prX^*$ | 20,160 | 48,384 | 18,816 |
| Investment cost (EUR) | $c\gamma K^*$ | 800 | 1288 | 672 |
| Feeding cost (EUR) | $\frac{\eta}{2}X^{*2}$ | 7,920 | 20,275 | 6,899 |
| Congestion cost (EUR) | $\frac{\theta}{2K}X^{*2}$ | 1,620 | 2,576 | 1,680 |
| Annual profit (EUR) | $\pi^*$ | 9,820 | 24,245 | 9,565 |

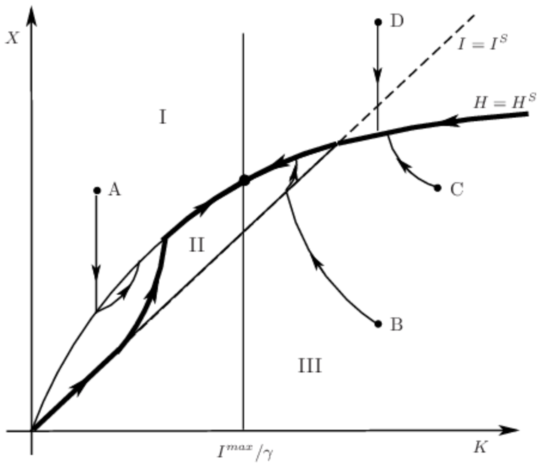
1 a)




1 b)




1 c)




1 a) Both controls singular near the steady state.

1 b): Approach paths with $I^S < I^{\max}$ near the equilibrium, but interior steady state.

1 c): Constrained steady state; $I^* < I^{\max}$.

**Figure 1: Optimal approach paths, unique steady state.**

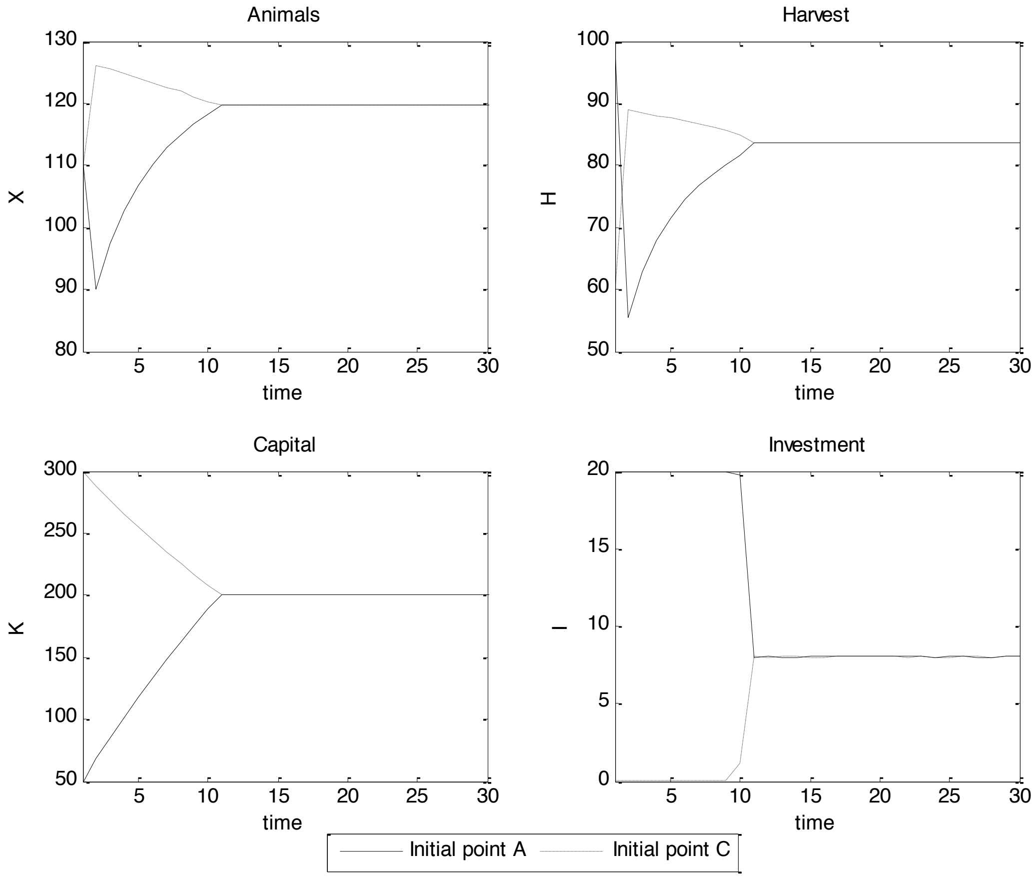



**Figure 2. Optimal approach paths baseline parameter values. Initial situation A ($K_0 = 50$, $X_0 = 110$) and C ($K_0 = 300, X_0 = 110$).**

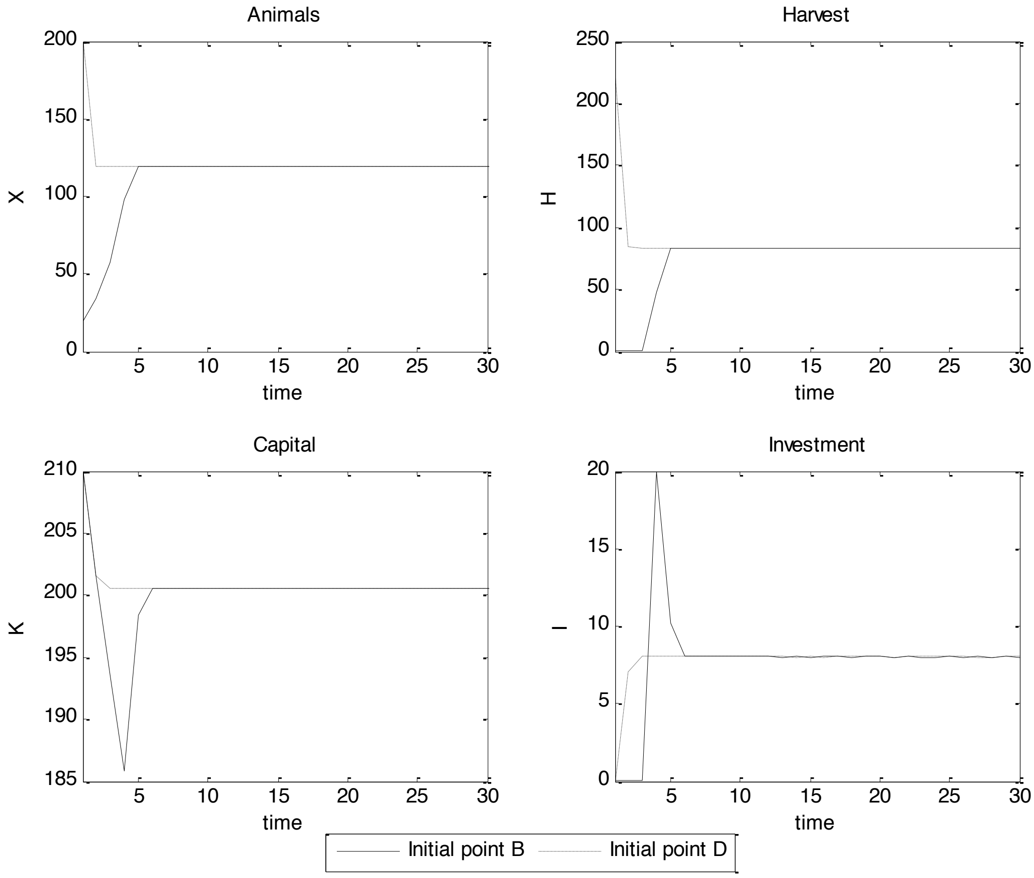



**Figure 3. Optimal approach paths baseline parameter values. Initial situation B ($K_0 = 210$, $X_0 = 20$) and D ($K_0 = 210, X_0 = 200$).**