

ISSN 1503-299X

# WORKING PAPER SERIES

No. 4/2007

## OPTIMAL PORTFOLIO CHOICE AND INVESTMENT IN EDUCATION

Snorre Lindset  
Egil Matsen

**Department of Economics**

---

 Norwegian University of Science and Technology  
N-7491 Trondheim, Norway  
[www.svt.ntnu.no/iso/wp/wp.htm](http://www.svt.ntnu.no/iso/wp/wp.htm)

# Optimal Portfolio Choice and Investment in Education

Snorre Lindset

Trondheim Business School

Hist avdeling TOH, 7004 Trondheim, Norway

and

Norwegian University of Science and Technology

Department of Industrial Economics

Phone: +47 73 55 99 78

Fax: +47 73 55 99 51

E-mail: [snorre.lindset@hist.no](mailto:snorre.lindset@hist.no)

Egil Matsen

Norwegian University of Science and Technology

Department of Economics

NTNU, 7491 Trondheim, Norway

Phone: +47 73 59 78 52

Fax: +47 73 59 19 40

E-mail: [egil.matsen@svt.ntnu.no](mailto:egil.matsen@svt.ntnu.no)

June 7, 2007

## **Abstract**

In this paper we analyze how an individual should optimally invest in her own human capital when she also has financial wealth. We treat the individual's option to take more education as expansion options and apply real option analysis. We characterize the individual's optimal consumption strategy and portfolio weights. The individual has a demand for hedging financial risk, labor income risk, and also wage level risk.

*Keywords and phrases:* Optimal portfolio choice, Investment in human capital, Hedging demand.

*JEL-classification:* C61, D14, G11, G13.

# 1 Introduction

In this paper we examine the decisions of a life-cycle investor that can invest in her own human capital, as well as financial assets. Two important characteristics of human capital investments are that they are *irreversible* and that they have *uncertain returns*.<sup>1</sup> Hence, investments opportunities in human capital are typical examples of projects with inherent option values. This is an important aspect of our analysis; the value of an individual's human wealth at a given point in time has two components:

1. The value of her human capital in place.
2. The value of her options to invest in more human capital at later points in time.

Standard models of saving and portfolio choice either ignore the existence of human capital or treat it as exogenous. Likewise, models of human capital accumulation and labor earnings over the life-cycle usually ignore portfolio choice. This is unfortunate, not only because human wealth is the most important asset for most young and many middle-aged individuals, but in particular because taking into account the interaction between human capital investments, labor income, savings, and portfolio choice can yield important insights into how such decisions are made (or should be made). Two exceptions are Bodie, Merton, and Samuelson (1992) and Veceira (2001), as they allow for endogenous labor supply. By varying her labor supply, the individual can affect her earnings (and thus the value of her human capital). However, these models do not analyze human capital investments *per se* and therefore do not take into account the option values inherent in the individual's wealth. Judd (2000)<sup>2</sup> and Williams (1978) solve static models of educational investments when the individual also can invest in financial assets. Although both authors recognize the irreversibility

---

<sup>1</sup>They also differ from many irreversible investments in physical capital in that the investor has a monopoly right to undertake the investment, because the property rights to human capital cannot be transferred. The latter point implies that that finiteness of life plays a central role in human capital investments (Blinder and Weiss (1976)).

<sup>2</sup>We should note that Judd's model incorporates moral hazard, while we treat the non-marketability of human capital as exogenous.

of human capital investments, the static framework precludes them from analyzing such investments by the real options approach.<sup>3</sup>

In this paper we derive the value of the human capital already in place for an individual. In addition we derive the value of the option to invest in more education. An individual's total wealth is the sum of the value of the human capital and the financial wealth. The non-marketability of the human capital causes a demand for hedging the risk associated with human capital. To this end we also characterize the individual's optimal consumption and portfolio strategy. In addition to the mean-variance tangency portfolio (see e.g., Merton (1969)), the individual now also hedges the risk from labor income and the risk from the rental price for human capital. The present paper is related to recent work by Saks and Shore (2005) on risk and career choice. While their focus is on the interaction between the type of education people choose and their portfolio choice, this paper is about the amount of education (in a broad sense) and portfolio choice.

The paper is organized as follows: In section 2 we lay out a life cycle model of human capital investments, savings, and portfolio choice. Assuming that risks to human capital investments are spanned by the traded assets of the economy, we demonstrate in section 3 how the individual's human wealth should be valued. We also analyze the implied profiles for earnings over the life-cycle. Given these profiles, we proceed to examine the individual's optimal savings and investment policies in section 4. The paper is concluded in section 5.

## 2 A life-cycle model

In this section we develop a life cycle model that will be used to explore the interactions between accumulation of human capital (i.e., education) and portfolio choice.

---

<sup>3</sup>Williams (1979) gives approximate solutions to an intertemporal model, but still ignores the option values inherent in the individual's wealth.

**Preferences and financial wealth** We study an individual that is assumed to live forever.<sup>4</sup> She derives utility from consumption only, and we treat her labor supply as fixed and exogenous. The individual's objective is to maximize

$$U_0 = E_0 \left[ \int_0^\infty e^{-\delta t} u(C(t)) dt \right], \quad (1)$$

where  $E_t$  is the conditional expectations operator,  $\delta$  is the rate of time preference,  $C(t)$  is consumption at time  $t$ , and  $u$  is an instantaneous utility function with standard properties.

At any time  $t$  the individual can invest in one riskless and one risky financial asset. The riskless asset has an instantaneous real return  $r$ , while the price process for the risky financial asset is given by

$$\frac{dP(t)}{P(t)} = \mu dt + \sigma dz(t). \quad (2)$$

Here,  $z(t)$  is a standard Wiener process, the constant  $\mu$  is the instantaneous expected rate of return on the asset, and  $\sigma$  is the instantaneous standard deviation of the return. Thus, the risky asset has a lognormal price distribution and normally distributed returns. Modelling financial investment opportunities in this manner is standard and dates back to Merton (1969).

Let  $F(t)$  denote the individual's financial wealth at time  $t$ , while  $\alpha(t)$  gives the share of *financial* wealth invested in the risky asset. Given (2), it is straightforward to show that the evolution of the financial wealth is given by

$$dF(t) = [(\alpha(t)(\mu - r) + r)F(t) - C(t)] dt + \alpha(t)F(t)\sigma dz(t) + dy(t), \quad (3)$$

where  $dy(t)$  is the flow of disposable labor income (to be defined below) at time  $t$ .

**Human capital and labor income** The individual has an initial stock of homogenous human capital (skills and knowledge)  $H(0) = H_0$ . There is a

---

<sup>4</sup>Infinite time horizon is necessary to obtain a closed-form solution to the option pricing problem considered in the next section. The modelling of human capital formation and labor income presented below give us reasonable life-cycle behavior despite the assumption of infinite horizon.

market in which the services of human capital are traded, and  $a(t)$  denotes the real rental price for a unit of human capital  $H(t)$  at time  $t$ . This price is taken as given by the individual. We assume that the rental price  $a(t)$  follows the geometric Brownian motion

$$\frac{da(t)}{a(t)} = \lambda dt + b\sigma dz(t), \quad (4)$$

where  $\lambda$  is a constant drift coefficient and  $b$  is a positive constant. As Bodie et al. (1992), we assume complete markets since the rental price of human capital is perfectly correlated with the risky financial asset. This is a necessary assumption to solve the option valuation problem in section 4.1 below. Human capital transforms into labor income through the Cobb-Douglas function

$$\hat{y}(t) = a(t)H(t)^\theta, \quad 0 < \theta \leq 1, \quad (5)$$

where  $\hat{y}(t)$  is realized labor income at time  $t$ . We notice that the marginal labor earnings product from human capital is  $a\theta H^{\theta-1}$ .

Next, we will assume that the individual can add to her human capital stock at any time  $t$  at a cost (in terms of the consumption good)  $k(t) = k_0 e^{\rho t}$  per unit added, with  $\rho > 0$  and  $k_0$  a constant. The cost of increasing the level of skills and knowledge is rising over time; human capital is only *partial* expandable.<sup>5</sup> It is this assumption that ensures reasonable life-cycle behavior despite our infinite time horizon setting.<sup>6</sup> We let  $dH(t) = Q(t)dt$  be the flow of acquired human capital at time  $t$ . For simplicity, we will ignore depreciation, so that  $Q(t)$  denotes net investment in human capital at time  $t$ . Then, by applying Ito's lemma to (5), we can write the flow of *disposable*<sup>7</sup> labor income as

$$dy(t) = a(t)H(t)^\theta [\lambda dt + b\sigma dz(t)] - \left[ k_0 e^{\rho t} - a(t)\theta H(t)^{\theta-1} \right] Q(t)dt. \quad (6)$$

---

<sup>5</sup>Dixit and Pindyck (1998) present a model with partial expandability of physical capital.

<sup>6</sup>As will be explained below, this model implies marginal cost of human capital investments shifts upward over time, while the marginal value will be constant. In classical human capital models with finite time horizon (see e.g., Ben-Porath (1967)), the marginal investment cost is constant over time, while the marginal value shifts down due to the finite horizon.

<sup>7</sup>Disposable for consumption and investment in financial assets, that is.

The first term on the right hand-side of (6) is the labor income flow delivered by the preexisting level of human capital, while the second term shows the net income from any investment in human capital at time  $t$ . The term in the last square brackets is *net* marginal investment cost at time  $t$ .<sup>8</sup> Note that net investment cost increases with time since the unit cost  $k$  increases exponentially over time.

### 3 Human wealth and labor income over the life-cycle

As argued in the introduction, the characteristics of human capital investments make them well suited to be analyzed by the real options approach.<sup>9</sup> In this section we use this approach to derive the value of the individual's human wealth, her optimal human capital investment policy, and the implied profile for earnings over the life-cycle. The next section incorporates this into the individual's broader savings and portfolio choice problem.

#### 3.1 Valuation of human wealth

At any time  $t$ , the value of the individual's human wealth consists of two components: the value of the human capital she already has, and the option value (evaluated at time  $t$ ) of investing in more human capital now or in the future. We will value each component in turn.

**Human capital in place** In general, the value at time  $t$  of the existing human capital is the expected discounted value of the (maximum) future wage income it can deliver. At time  $t$ , the individual's stock of human capital is  $H(t)$ . If she makes no new investments in human capital, this stock will be constant over time. However, the rental price will fluctuate, so at time  $\tau$ ,  $t \leq \tau$ , labor earnings are  $y(\tau) = a(\tau)H(t)^\theta$ . Combining (4) and

---

<sup>8</sup>The word net is important here. As the model is set up, acquiring a marginal unit of human capital gives an asset with a certain value (to be determined later in this section), but it also gives immediate income  $a\theta H^{\theta-1}$ . The net investment cost of a marginal unit is thus the gross cost  $k$  less the instantaneous income provided by the unit.

<sup>9</sup>The authoritative reference to this approach is of course Dixit and Pindyck (1993).

(5), we have that

$$dy/y = \lambda dt + b\sigma dz. \quad (7)$$

We can then follow Bodie et al. (1992) (see their section 4) to demonstrate that the value of the individual's human capital in place is given by

$$V(H; a, t) = \frac{y(t)}{r + b(\mu - r) - \lambda} = \frac{a(t)H(t)^\theta}{r + b(\mu - r) - \lambda}, \quad (8)$$

where we assume that the denominator is positive. We can notice that the marginal value of acquired human capital at time  $t$  is

$$v(H; a, t) \equiv \frac{\partial V}{\partial H} = \frac{\theta a(t)H(t)^{\theta-1}}{r + b(\mu - r) - \lambda}, \quad (9)$$

a concave function in  $H$ .

**Option value of investing in more human capital** We now analyze the individual's options to invest in additional human capital. Denote the value of these options by  $G(H; a, t)$ . Since we have assumed complete markets, we can follow Dixit and Pindyck (1993) and Dixit and Pindyck (1998) and set up a risk free portfolio to determine  $G$ . Suppose that we hold one unit of the (portfolio of) expansion options and sell short  $m$  units of the spanning asset  $n$ . In the appendix, we demonstrate that this gives the following differential equation for the value of the *marginal* expansion option  $g(H; a, t) \equiv -\partial G/\partial H$ :<sup>10</sup>

$$\frac{1}{2}(b\sigma)^2 a^2 \frac{\partial^2 g}{\partial a^2} + [\lambda - b(\mu - r)] a \frac{\partial g}{\partial a} - rg + \frac{\partial g}{\partial t} = 0. \quad (10)$$

This partial differential equation is subject to the four boundary conditions

$$g(H; 0, t) = 0 \quad (11)$$

$$g(H; a^*, t) = v(H; a^*, t) - I(H; a^*, t) \quad (12)$$

$$\frac{\partial g(H; a^*, t)}{\partial a} = \frac{\partial v(H; a^*, t)}{\partial a} - \frac{\partial I(H; a^*, t)}{\partial a} \quad (13)$$

$$\lim_{t \rightarrow \infty} g(H; a, t) = 0. \quad (14)$$

If  $a$  hits zero, it will stay at zero and the opportunity to invest in human capital will be worthless; hence the first boundary condition. Equations (12)

---

<sup>10</sup>An increase in  $H$  means exercising some of the future expansion options, so  $\partial G/\partial H$  must be negative.

and (13) are the value matching and smooth pasting conditions, respectively (see e.g., Dixit and Pindyck (1993)). The former says that at the value  $a^*$  where it is optimal to exercise the marginal option, the individual receives a net payoff equal to the present value of labor income it delivers minus the net marginal cost. The smooth pasting condition requires that  $g(H; a, t)$  is continuous and smooth at the critical exercise point  $a^*$ ; if not one could do better by exercising at a different point. Finally, boundary condition (14) says that the value of the option to invest in a marginal unit of human capital approaches zero as time passes by. This follows since the cost of exercising the option (the ‘strike price’) is increasing exponentially with time.

We demonstrate in the appendix that the solution to (10) is given by

$$g(H; a, t) = B(H)a(t)^{\beta_1}e^{-qt}, \quad (15)$$

where  $B(H)$  and  $q$  are parameters to be determined, while

$$\beta_1 = \frac{1}{2} - \frac{\lambda - b(\mu - r)}{(b\sigma)^2} + \sqrt{\left(\frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \frac{1}{2}\right)^2 + \frac{2(r + q)}{(b\sigma)^2}} > 1. \quad (16)$$

Given (15), we can use (12) and (13) to solve for the critical exercise value  $a^*$

$$a^*(H, t) = \frac{\beta_1}{\beta_1 - 1} \frac{[r + b(\mu - r) - \lambda]k_0e^{\rho t}}{\theta H(t)^{\theta-1}}. \quad (17)$$

The product  $[r + b(\mu - r) - \lambda]k_0e^{\rho t}$  in (17) can be interpreted as the instantaneous flow cost of increasing the human capital stock by a marginal unit at time  $t$ . Equation (17) illustrates that the value of the current marginal labor earnings product  $a\theta H^{\theta-1}$  must be a multiple  $\beta_1/(1 - \beta_1) > 1$  of this flow cost to trigger investment. The rental price of human capital must cover the full cost of investing: the direct flow cost plus the opportunity cost of investing now instead of later. We notice that the human capital rental price necessary to induce investment is increasing with time, due to rising costs over time, and with the level of human capital, due to decreasing marginal labor earnings product.

Substituting (17) into (13) gives the following expression for  $B(H)$

$$B(H) = (\beta_1 - 1)^{\beta_1-1} \left( \frac{\theta H^{\theta-1}}{[r + b(\mu - r) - \lambda]\beta_1} \right)^{\beta_1} k_0^{1-\beta_1} e^{[q - \rho(\beta_1-1)]t}. \quad (18)$$

Since  $B(H)$  does not depend on  $t$ , we have from (18) that  $q = \rho[\beta_1(q) - 1]$ . Substituting equation (16) for  $\beta_1(q)$  and solving for  $q$  gives

$$q = \rho \left[ -\frac{1}{2} - \frac{[r + b(\mu - r) - \lambda] - \rho}{(b\sigma)^2} + \sqrt{\left( \frac{[r + b(\mu - r) - \lambda] - \rho}{(b\sigma)^2} + \frac{1}{2} \right)^2 + \frac{2(\mu - \lambda)}{(b\sigma)^2}} \right] > 0.$$

Now, (15) gives the value of the option to produce a marginal unit of human capital at time  $t$ . In principle, the individual could produce an infinite amount of human capital, but we recall that the marginal labor earnings product is decreasing in the preexisting level of human capital, and this effect enters through the ‘constant’ in (18). Using (15), the total value of the expansion options, evaluated at time  $t$  is

$$\begin{aligned} G(H; a, t) &= \int_{H(t)}^{\infty} B(H)a(t)^{\beta_1} e^{-qt} dH \\ &= \left( \frac{\theta a(t)}{[r + b(\mu - r) - \lambda]\beta_1} \right)^{\beta_1} \left( \frac{\beta_1 - 1}{k_0} \right)^{\beta_1 - 1} e^{-qt} \int_{H(t)}^{\infty} h^{(\theta - 1)\beta_1} dh. \end{aligned}$$

A *sufficient* condition for the latter integral to converge is that  $\beta_1 > 1/(1 - \theta)$ . That is, if the increase in labor income from more human capital decreases sufficiently fast ( $\theta$  is sufficiently less than 1), we can be sure that the individual’s expansion options have finite value for all  $t$ . We will throughout the paper assume that this convergence condition holds. It then follows that the total value of the expansion options is given by

$$G(H; a, t) = \left( \frac{\beta_1 - 1}{k_0} \right)^{\beta_1 - 1} \frac{H(t)^{1 - (1 - \theta)\beta_1}}{(1 - \theta)\beta_1 - 1} \left( \frac{\theta a(t)}{[r + b(\mu - r) - \lambda]\beta_1} \right)^{\beta_1} e^{-qt}. \quad (19)$$

The value of the expansion options is increasing in the rental price, decreasing in the stock of human capital, and will approach zero as time passes by since the cost of investing in skills and knowledge rises exponentially with time.

### 3.2 Investment in human capital and labor earnings

The option valuation in the previous subsection is helpful in analyzing the optimal human capital investment policy for the individual. The function

$a^*(H, t)$  implicitly defines the optimal human capital level at every instant. If, at time  $t$ ,  $a$  and  $H$  are such that  $a(t) > a^*(H, t)$ , the individual should invest in human capital until  $a^* = a(t)$ . Equivalently, we can rearrange (17) in terms of  $H^*(a, t)$ , and express the optimal level of human capital at time  $t$  as

$$H^*(a, t) = \left( \frac{\beta_1 - 1}{\beta_1} \frac{a(t)\theta}{[r + b(\mu - r) - \lambda] k_0 e^{\rho t}} \right)^{\frac{1}{1-\theta}}. \quad (20)$$

Thus, if  $H^* > H(t)$ , the individual would add to her human capital stock, whereas no investment would be undertaken if  $H^* \leq H(t)$ . Summarizing, the optimal human capital investment policy can be written as

$$\begin{aligned} Q^*(a, t; H) &= H^*(a, t) - H(t), & \text{if } a^* < a(t), \\ &= 0 & \text{if } a^* \geq a(t). \end{aligned} \quad (21)$$

Given  $a^* > a(t)$ , the amount of human capital investment is increasing in the rental price  $a$  and decreasing with time  $t$ .

Combining (6) with the optimal investment policy in (21) we have that the *optimal* disposable labor income flow is given by

$$\begin{aligned} dy^* &= aH^\theta (\lambda dt + b\sigma dz) \\ &+ j \left[ \left( \left( \frac{\beta_1 - 1}{\beta_1} \frac{a\theta}{[r + b(\mu - r) - \lambda] k_0 e^{\rho t}} \right)^{\frac{1}{1-\theta}} - H \right) (a\theta H^{\theta-1} - k_0 e^{\rho t}) \right] dt, \end{aligned} \quad (22)$$

where  $j$  is an indicator function with  $j = 1$  if  $a^* \geq a(t)$ , and  $j = 0$  otherwise.

## 4 Consumption and portfolio choice

We now turn to the consumption and portfolio decision of the individual. Formally, the problem is to choose paths for  $C(t)$ ,  $Q(t)$ , and  $\alpha(t)$  to maximize the expected utility in (1), subject to the evolution of the state variables. We have already found the optimal solution for  $Q(t)$  (equation (21)). At this stage we can incorporate this solution in the individual's maximization problem, upon which the problem is reduced to a pure consumption/portfolio choice problem.

The indirect utility function for this problem is defined by

$$J(F, a, H, t) = \max_{\{\alpha(\tau), C(\tau)\}} E_t \left[ \int_t^\infty e^{-\delta\tau} u(C(\tau)) d\tau \right],$$

and the maximization is subject to the budget equation

$$dF = [(\alpha(\mu - r) + r)F - C] dt + \alpha F \sigma dz \\ + aH^\theta [\lambda dt + b\sigma dz] + jQ^*(H; a, t) \left( a\theta H^{\theta-1} - k_0 e^{\rho t} \right) dt,$$

equation (4), the (optimal) evolution of human capital,  $dH^* = jQ^* dt$ , and the current values  $F(t)$ ,  $a(t)$ , and  $C(t)$ . The Hamilton-Jacobi-Bellman equation is

$$0 = \max_{\{\alpha, C\}} \{u(C)e^{-\delta t} + J_F [(\alpha(\mu - r) + r)F - C + aH^\theta \lambda + jQ^*(a\theta H^{\theta-1} - k_0 e^{\rho t})] \\ + J_t + \frac{1}{2} J_{FF} [(\alpha F \sigma)^2 + 2\alpha F a H^\theta b \sigma^2 + (a H^\theta b \sigma)^2] + J_a a \lambda + \frac{1}{2} J_{aa} (ab\sigma)^2 \\ + J_H jQ^* + J_{Fa} ab\sigma^2 (\alpha F + baH^\theta)\},$$

where subscripts denote partial derivatives with respect to the designated variables. The resulting first-order conditions are

$$u_C(C)e^{-\delta t} = J_F$$

and

$$J_F(\mu - r) + J_{FF}\sigma^2(\alpha F + baH^\theta) + J_{Fa}ab\sigma^2 = 0.$$

We can now determine the optimal consumption function and the optimal portfolio choice, i.e.,

$$C^*(F, a, H, t) = u_C^{-1}(J_F e^{\delta t}) \quad (23)$$

and

$$(\alpha F)^*(F, a, H, t) = -\frac{J_F}{J_{FF}} \frac{\mu - r}{\sigma^2} - by(t) - \frac{J_{Fa}}{J_{FF}} ba(t), \quad (24)$$

where we have used  $\hat{y}(t) = a(t)H(t)^\theta$ .

We recognize (23) as the envelope condition. At the optimum an extra unit of consumption is as valuable to the investor as an extra unit of wealth to finance future consumption. From (24) we see that the optimal portfolio can be decomposed into three terms (Svensson and Werner (1993)):

1. The first term on the right hand-side is usual mean-variance tangency portfolio. It is the reciprocal of the coefficient of absolute risk aversion  $-J_F/J_{FF}$  times the expected excess return to the variance of the risky financial asset.

2. The second term can be labeled the “labor income hedge portfolio”. It gives the substitution away from the risky asset needed to (perfectly) hedge the variability of labor income.
3. The third term gives the adjustment necessary to hedge the uncertainty associated with the state variable  $a$ . This term can thus be labeled the “human capital rental price hedge portfolio”, where  $J_{Fa}/J_{FF}$  is the ratio of the absolute aversion of rental price risk to the absolute aversion to (financial) wealth risk.

Substitution of (23) and (24) back into the Hamilton-Jacobi-Bellman equation delivers a second-order partial differential equation (PDE) for the value function  $J(F, a, H, t)$ . It is generally not possible to find an analytical solution for this PDE, so one has to resort to numerical solutions. Those solutions will not, however, affect the basic economic intuition for the problem we have analyzed in this paper.

## 5 Conclusions

We have analyzed how an individual should optimally invest in education, consume, and construct her financial portfolio. The individual’s wealth has been divided into human wealth and financial wealth. The human wealth has further been divided into human capital already in place and value of the option to invest in more education. We show that there exist a wage level  $a^*$ , i.e., rental price for human capital, that triggers the individual to invest in more education. When taking human capital into account, we show that the individual’s optimal portfolio weights take the human capital into account. The individual now also hedges the labor income and the uncertainty in the wage level.

## A Derivation of selected equations

### A.1 Equation (10)

The portfolio of expansion options and  $m$  units of the risky financial asset costs  $G(H; a, t) - mP(t)$  to buy. Suppose that the options go unexercised

at time  $t$ . The portfolio pays no dividend, but by Ito's lemma, (2) and (4), instantaneous capital gains are

$$\begin{aligned} dG - mdP &= \left( \frac{\partial G}{\partial t} + \lambda a \frac{\partial G}{\partial a} + \frac{1}{2}(b\sigma)^2 a^2 \frac{\partial^2 G}{\partial a^2} - m\mu P \right) dt \\ &\quad + \left( b\sigma a \frac{\partial G}{\partial a} - m\sigma P \right) dz. \end{aligned}$$

By choosing  $m = \frac{ba}{P} \frac{\partial G}{\partial a}$  at every instant, the portfolio will be risk free. In the absence of arbitrage we must accordingly have

$$\begin{aligned} &\left( \frac{\partial G}{\partial t} + \lambda a \frac{\partial G}{\partial a} + \frac{1}{2}(b\sigma)^2 a^2 \frac{\partial^2 G}{\partial a^2} - \frac{ba}{P} \frac{\partial G}{\partial a} \mu P \right) dt \\ &= r \left( G - \frac{ba}{P} \frac{\partial G}{\partial a} P \right) dt. \end{aligned}$$

Finally, we differentiate this expression with respect to  $H$ , use the definition  $g(H; a, t) = -\partial G / \partial H$ , and rearrange to obtain (10).

## A.2 Equation (15)

By substitution, we can readily confirm that the function  $g = Ba(t)^\beta e^{-qt}$  satisfies (10), provided that  $\beta$  is a root of

$$\frac{1}{2}(b\sigma)^2 \beta(\beta - 1) + [\lambda - b(\mu - r)]\beta - r - q = 0.$$

The first root  $\beta_1$  is given in equation (16), while the second is

$$\beta_2 = \frac{1}{2} - \frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \sqrt{\left( \frac{\lambda - b(\mu - r)}{(b\sigma)^2} - \frac{1}{2} \right)^2 + \frac{2(r + q)}{(b\sigma)^2}} < 0.$$

The general solution to (10) is thus

$$g(H; a, t) = Ba^{\beta_1} e^{-qt} + \tilde{B}a^{\beta_2} e^{-qt},$$

but since  $\beta_2 < 0$  the first boundary condition implies that  $\tilde{B} = 0$ , and we are left with (15).

## References

Ben-Porath, Y. (1967). "The Production of Human Capital and the Life Cycle of Earnings", *Journal of Political Economy*, 75(4), 352–365.

- Blinder, A. and Weiss, Y. (1976). “Human capital and labor supply: A synthesis”, *Journal of Political Economy*, 84(3), 449–472.
- Bodie, Z., Merton, R. C., and Samuelson, W. F. (1992). “Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model”, *Journal of Economic Dynamics and Control*, 16(3-4), 427–449.
- Dixit, A. and Pindyck, R. (1998). “Expandability, reversibility and optimal capacity choice”, NBER Working Paper No. 6373.
- Dixit, A. K. and Pindyck, R. S. (1993). *Investment under Uncertainty*. Princeton University Press, New Jersey.
- Judd, K. (2000). “Is Education as Good as Gold? A Portfolio Analysis of Human Capital Investment”, Working Paper, Stanford University.
- Merton, R. C. (1969). “Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case”, *Review of Economics and Statistics*, 51, 247–257.
- Saks, R. E. and Shore, S. H. (2005). “Risk and Career Choice”, *Advances in Economic Analysis & Policy*, 5(1), Article 7.
- Svensson, L. and Werner, I. (1993). “Nontraded assets in incomplete markets: Pricing and portfolio choice”, *European Economic Review*, 37(5), 1149–1168.
- Viceira, L. M. (2001). “Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income”, *Journal of Finance*, 56(2), 433–470.
- Williams, J. T. (1978). “Risk, Human Capital, and the Investor’s Portfolio”, *Journal of Business*, 51(1), 65–89.
- Williams, J. T. (1979). “Uncertainty and the Accumulation of Human Capital Over the Life Cycle”, *Journal of Business*, 52(4), 521–548.