

Chapter 5. Markov Methods

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Learning Objectives

The main learning objectives associated with these slides are to:

- ▶ Introduce Markov methods for system reliability analysis
- ▶ Explain different ways to solve Markov equations, including:
 - Time dependent
 - Steady state
 - Mean time to first failure (MTTF_S)

The slides include topics from Chapter 5 in **Reliability of Safety-Critical Systems: Theory and Applications**. DOI:10.1002/9781118776353.

Outline of Presentation

- 1 Introduction
- 2 Markov Basics
- 3 Solving the Markov Process
- 4 Time Dependent Solution
- 5 Steady-State Solution
- 6 $MTTF_S$

Markov Approach in Brief

Some keywords:

- ▶ Suitable for dynamic systems
- ▶ Must satisfy the Markov properties
- ▶ Can model system states, beyond failure states
- ▶ Can be used to model steady state and time-dependent probabilities
- ▶ Can also be used to model mean time to first failure (MTTF_S)

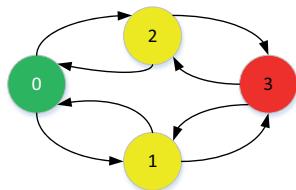


Figure: Russian mathematician Andrei Markov (1856-1922)

States and State Transitions

Consider a parallel structure of two components. Each component is assumed to have two states, a functioning state and a failed state. The structure has therefore $2^2 = 4$ possible states, and the state space is $\mathcal{X} = \{0, 1, 2, 3\}$

State	Component 1	Component 2
0	Functioning	Functioning
1	Functioning	Failed
2	Failed	Functioning
3	Failed	Failed



State Space

Let $X(t)$ denote the state of the system at time t .

The state space is the set of all the possible system states. In this book we number the states by integers from 0 to r . The state space is therefore

$$\mathcal{X} = \{0, 1, 2, \dots, r\}$$

Let $P_i(t) = \Pr(X(t) = i)$ be the probability that the system is in state i at time t .

The state probability distribution is denoted

$$\mathbf{P}(t) = (P_0(t), P_1(t), \dots, P_r(t))$$

Markov Property

Assume that the process is in state i at time s , that is, $X(s) = i$. The conditional probability that the process will be in state j at time $t + s$ is

$$\Pr(X(s + t) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s)$$

where $\{x(u), 0 \leq u < s\}$ denotes the “history” of the process up to, but not including, time s

The process is said to have the *Markov property* if

$$\Pr(X(s + t) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < s)$$

$$= \Pr(X(s + t) = j \mid X(s) = i)$$

for all possible $x(u)$, $0 \leq u < s$

Markov Property

So, what is Markov property (in easy words)

- ▶ The probability of being in a specific state state at a future time t only depends ONLY on the state of the system right now (s) state, and NOT at all about the states the system has had before (before s).
- ▶ A “memorylessness” process, where the next state of the process depends only on the previous state and not the sequence of states.
- ▶ Your next move is purely random: Every time you are in a state, you flip a coin or a dice to decide where to go next.

Markov Process

A ‘continuous time’ stochastic process that fulfills the Markov property is called a *Markov process*.

We will further assume that the Markov process for all i, j in \mathcal{X} fulfills

$$\Pr(X(s+t) = j \mid X(s) = i) = \Pr(X(t) = j \mid X(0) = i)$$

for all $s, t \geq 0$

which says that the probability of a transition from state i to state j does not depend on the global time and only depends on the time interval available for the transition.

A process with this property is known as a process with *stationary transition probabilities*, or as a *time-homogeneous process*.

Transition Probabilities

The transition probabilities of the Markov process

$$P_{ij}(t) = \Pr(X(t) = j \mid X(0) = i)$$

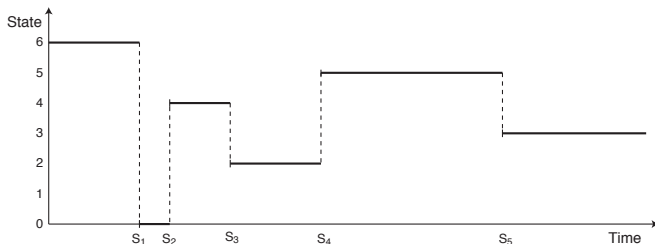
may be arranged as a matrix

$$\mathbb{P}(t) = \begin{pmatrix} P_{00}(t) & P_{01}(t) & \cdots & P_{0r}(t) \\ P_{10}(t) & P_{11}(t) & \cdots & P_{1r}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{r0}(t) & P_{r1}(t) & \cdots & P_{rr}(t) \end{pmatrix}$$

When a process is in state i at time 0, it must either be in state i at time t or have made a transition to a different state. We must therefore have

$$\sum_{j=0}^r P_{ij}(t) = 1$$

Trajectory



Let $0 = S_0 \leq S_1 \leq S_2 \leq \dots$ be the times at which transitions occur, and let $T_i = S_{i+1} - S_i$ be the i th interoccurrence time, or *sojourn time*, for $i = 1, 2, \dots$

A possible trajectory of a Markov process is illustrated above.

We define S_i such that transition i takes place immediately before S_i , in which case the trajectory of the process is continuous from the right.

Time in a State

A Markov process enters state i at time 0, such that $X(0) = i$. Let \tilde{T}_i be the sojourn time in state i . [Note that T_i denotes the i th interoccurrence time, while \tilde{T}_i is the time spent during a visit to state i .]

We want to find $\Pr(\tilde{T}_i > t)$. We observe that the process is still in state i at time s , that is, $\tilde{T}_i > s$, and are interested in finding the probability that it will remain in state i for t time units more. We hence want to find $\Pr(\tilde{T}_i > t + s \mid \tilde{T}_i > s)$.

Since the process has the Markov property, the probability for the process to stay for t more time units is determined only by the current state i . The fact that the process has been staying there for s time units is therefore irrelevant. Thus

$$\Pr(\tilde{T}_i > t + s \mid \tilde{T}_i > s) = \Pr(\tilde{T}_i > t) \quad \text{for } s, t \geq 0$$

Hence \tilde{T}_i is *memoryless* and must be exponentially distributed.

Sojourn Times

From the previous frame we can conclude that the sojourn times T_1, T_2, \dots are independent and exponentially distributed. The independence follows from the Markov property.

Constructing the Markov Process

We may construct a Markov process as a stochastic process having the properties that each time it enters a state i :

1. The amount of time \tilde{T}_i the process spends in state i before making a transition into a different state is exponentially distributed with rate, say α_i .
2. When the process leaves state i , it will next enter state j with some probability P_{ij} , where $\sum_{\substack{j=0 \\ j \neq i}}^r P_{ij} = 1$.

The mean sojourn time in state i is therefore

$$E(\tilde{T}_i) = \frac{1}{\alpha_i}$$

Transition Rates

Let a_{ij} be defined by

$$a_{ij} = \alpha_i \cdot P_{ij} \quad \text{for all } i \neq j$$

Since α_i is the rate at which the process leaves state i and P_{ij} is the probability that it goes to state j , it follows that a_{ij} is the rate when in state i that the process makes a transition into state j . We call a_{ij} the *transition rate* from i to j .

Since $\sum_{j \neq i} P_{ij} = 1$, it that

$$\alpha_i = \sum_{\substack{j=0 \\ j \neq i}}^r a_{ij}$$

Transition Rates (2)

Let T_{ij} be the time the process spends in state i before entering into state j ($j \neq i$). The time T_{ij} is exponentially distributed with rate a_{ij} .

Consider a short time interval Δt . Since T_{ij} and \tilde{T}_i are exponentially distributed, we have that

$$P_{ii}(\Delta t) = \Pr(\tilde{T}_i > \Delta t) = e^{-\alpha_i \Delta t} \approx 1 - \alpha_i \Delta t$$

$$P_{ij}(\Delta t) = \Pr(T_{ij} \leq \Delta t) = 1 - e^{-a_{ij} \Delta t} \approx a_{ij} \Delta t$$

when Δt is “small”. We therefore have that

$$\lim_{\Delta t \rightarrow 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Pr(\tilde{T}_i < \Delta t)}{\Delta t} = \alpha_i$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_{ij}(\Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Pr(T_{ij} < \Delta t)}{\Delta t} = a_{ij} \quad \text{for } i \neq j$$

Transition Rate Matrix

We can deduce α_i and P_{ij} when we know a_{ij} for all i, j , and may therefore define the Markov process by specifying (i) the state space \mathcal{X} and (ii) the transition rates a_{ij} for all $i \neq j$ in \mathcal{X} . The second definition is often more natural and will be our main approach in the following.

We may arrange the transition rates a_{ij} as a matrix:

$$\mathbb{A} = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0r} \\ a_{10} & a_{11} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r0} & a_{r1} & \cdots & a_{rr} \end{pmatrix}$$

where the diagonal elements are

$$a_{ii} = -\alpha_i = - \sum_{\substack{j=0 \\ j \neq i}}^r a_{ij}$$

Departure Rates

Observe that the entries of row i of the transition rate matrix \mathbb{A} are the transition rates out of state i (for $j \neq i$). We will call them *departure rates* from state i .

The rate $-a_{ii} = \alpha_i$ is the *total* departure rate from state i . Note that:

- ▶ α_i is also the rate of staying in state i
- ▶ a_{ij} is the rate of leaving from state i to state j

The sum of the entries in row i is always equal to 0, for all $i \in \mathcal{X}$.

Kolmogorov Differential Equations

To find $P_{ij}(t)$ we start by considering the Chapman-Kolmogorov equations

$$P_{ij}(t + \Delta t) = \sum_{k=0}^r P_{ik}(t)P_{kj}(\Delta t)$$

We consider

$$P_{ij}(t + \Delta t) - P_{ij}(t) = \sum_{\substack{k=0 \\ k \neq j}}^r P_{ik}(t)P_{kj}(\Delta t) - [1 - P_{jj}(\Delta t)]P_{ij}(t)$$

By dividing by Δt and then taking the limit as $\Delta t \rightarrow 0$, we obtain

$$\lim_{\Delta t \rightarrow 0} \frac{P_{ij}(t + \Delta t) - P_{ij}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left[\sum_{\substack{k=0 \\ k \neq j}}^r P_{ik}(t) \frac{P_{kj}(\Delta t)}{\Delta t} - \frac{1 - P_{jj}(\Delta t)}{\Delta t} P_{ij}(t) \right]$$

Kolmogorov Differential Equations (2)

Since the summing index is finite, we may interchange the limit and summation and obtain

$$\dot{P}_{ij}(t) = \sum_{\substack{k=0 \\ k \neq j}}^r a_{kj} P_{ik}(t) - \alpha_j P_{ij}(t) = \sum_{k=0}^r a_{kj} P_{ik}(t)$$

where, as before, $a_{jj} = -\alpha_j$. The differential equations above are known as the Kolmogorov *forward equations*.

If the **marked t and Δt** are interchanged, you can develop the Kolmogorov backward equations.

Kolmogorov Differential Equations (3)

The **Kolmogorov forward equations** may be written in matrix format as:

$$\mathbb{P}(t) \cdot \mathbb{A} = \dot{\mathbb{P}}(t)$$

which we can also write as

$$\dot{\mathbb{P}}(t) = \mathbb{P}(t) \cdot \mathbb{A}$$

In the following, we will use the **Kolmogorov forward** equation, but in principle we could choose either one.

What to Solve for

There are three main ways we will solve the Markov equations:

1. For calculating the time dependent probabilities
2. For calculating the steady state probabilities
3. For calculating the mean time to first failure ($MTTF_S$)

In the following, we will present them all.

Steps

The main benefit of time-dependent probabilities is that we can study how the probabilities (e.g., availability and unavailability) change with time.

To find the time-dependent probabilities for a specific transition model, we have to:

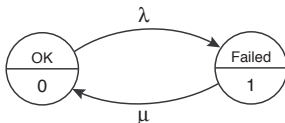
1. Set up the transition matrix
2. Set up the differential equations
3. Solve these equations, either by hand or using tools like e.g., MAPLE.

Example

Consider a single system item in continuous use and which is repaired if it fails. The system states are:

State	State description
0	The item is functioning
1	The item has failed

The corresponding state diagram is:



Example (cont.)

- ▶ Step 1:
Setting up the transition matrix is

$$\mathbb{A}_t = \begin{pmatrix} -\lambda + \mu & \lambda \\ \mu & -\mu \end{pmatrix}$$

- ▶ Step 2:
Setting up the differential equations:

$$-\lambda P_0(t) + \mu P_1(t) = \dot{P}_0(t)$$

$$\lambda P_0(t) - \mu P_1(t) = \dot{P}_1(t)$$

Example (cont.)

- ▶ Step 3:
Solving for $P_0(t)$ and $P_1(t)$.

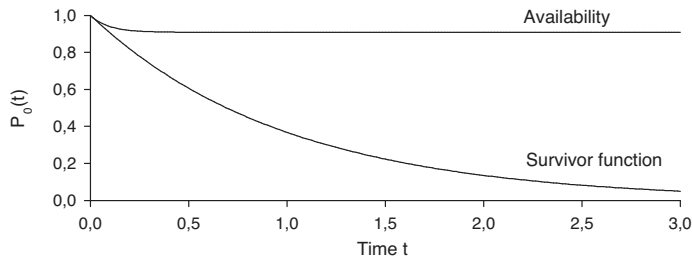
You may use dsolve-function in MAPLE to solve with the initial conditions that $P_0(0) = 1$ and $P_1(0) = 0$ and find that:

$$P_0(t) = \frac{\mu}{\mu + \lambda} + \frac{\mu}{\mu + \lambda} e^{-(\lambda + \mu)t}$$

$$P_1(t) = \frac{\lambda}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t}$$

Example (cont.)

The following plots survivability and availability, using $\lambda = 1$ and $\mu = 10$.



Steps

The Markov models often enter a steady-state after few hours, typically 2-3 times the mean repair time. In this case, it may be of more interest to study steady-state probabilities rather than time-dependent.

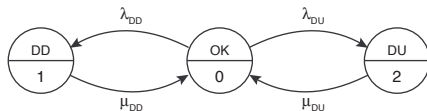
To find the steady-state probabilities for a specific transition model, we have to:

1. Set up the transition matrix (as we did for time dependent solution)
2. Set up the *steady state equations*
3. Solve these equations, either by hand or using tools like e.g., MAPLE, bearing in mind that $\sum P_i = 1$.

Example: Single System

Consider a single system that may fail due to DU or DD failures. The system states are:

State	State description
0	The channel is functioning (no DU or DD failures)
1	The channel has a DD fault
2	The channel has a DU fault



Example: Single System (cont.)

Step 1:

Set up transition matrix:

$$\mathbb{A} = \begin{pmatrix} -(\lambda_{DD} + \lambda_{DU}) & \lambda_{DD} & \lambda_{DU} \\ \mu_{DU} & -\mu_{DU} & 0 \\ \mu_{DU} & 0 & -\mu_{DU} \end{pmatrix}$$

Parameter	Description	Comments
λ_{DU}	Dangerous undetected (DU) failure rate	
λ_{DD}	Dangerous undetected (DD) failure rate	
μ_{DU}	Repair rate of DU failures	$1/(\frac{\tau}{n-k+2} + MRT)$
μ_{DD}	Repair rate of DD failures	$1/MTTR$

Example: Single System (cont.)

Step 2:

Set up the steady-state equations, i.e. $\mathbf{P}\mathbf{A} = \mathbf{0}$

- ▶ Select $r - 1$ (in this case two) of the r (in this case three) equations, preferably the ones with most zeros, and add the equation

$$P_0 + P_1 + P_2 = 1.$$

The equations become

$$P_0 + P_1 + P_2 = 1$$

$$\lambda_{DD}P_0 - \mu_{DD}P_1 = 0$$

$$\lambda_{DU}P_0 - \mu_{DU}P_2 = 0$$

Example: Single System (cont.)

Step 3:

Solve for the steady state probabilities, in this case P_0 , P_1 , and P_2 :

- ▶ By hand-calculations or e.g. MAPLE, we find that:

$$P_0 = \frac{1}{\frac{\lambda_{DD}}{\mu_{DD}} + \frac{\lambda_{DU}}{\mu_{DU}} + 1}$$

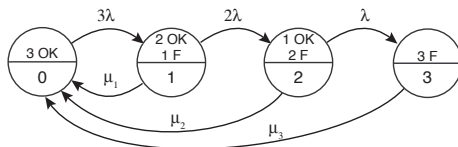
$$P_1 = \frac{\lambda_{DD}}{\mu_{DD}} P_0$$

$$P_2 = \frac{\lambda_{DU}}{\mu_{DU}} P_0$$

Example: 2oo3 system

Consider a subsystem of three identical and independent channels with failure rate λ voted 2oo3. Since the channels are identical, it suffices to define four states.

State	State description
0	Three channels are functioning
1	Two channels are functioning and one is failed
2	One channel is functioning and two are failed
3	Three channels are failed



Example: 2003 System(cont.)

The state transition matrix becomes:

$$\mathbb{A} = \begin{pmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu_1 & -(\mu_1 + 2\lambda) & 2\lambda & 0 \\ \mu_2 & 0 & -(\mu_2 + \lambda) & \lambda \\ \mu_3 & 0 & 0 & -\mu_3 \end{pmatrix}$$

The steady-state equations are (replacing one of the most favorable of the equations by $P_0 + P_1 + P_2 + P_3 = 1$):

$$[P_0, P_1, P_2, P_3] \begin{pmatrix} 1 & 3\lambda & 0 & 0 \\ 1 & -(\mu_1 + 2\lambda) & 2\lambda & 0 \\ 1 & 0 & -(\mu_2 + \lambda) & \lambda \\ 1 & 0 & 0 & -\mu_3 \end{pmatrix} = [1, 0, 0, 0]$$

Example: 2003 System (cont.)

Solving for steady-state probabilities using MAPLE gives the following solutions:

$$P_0 = \frac{\mu_3(2\lambda^2 + 2\lambda\mu_2 + \mu_1\mu_2 + \lambda\mu_1)}{6\lambda^3 + 11\lambda^2\mu_3 + 5\lambda\mu_2\mu_3 + \mu_1\mu_2\mu_3 + \lambda\mu_1\mu_3} \approx 0.9988$$

$$P_1 = \frac{3\lambda\mu_3(\lambda + \mu_2)}{6\lambda^3 + 11\lambda^2\mu_3 + 5\lambda\mu_2\mu_3 + \mu_1\mu_2\mu_3 + \lambda\mu_1\mu_3} \approx 1.12 \cdot 10^{-3}$$

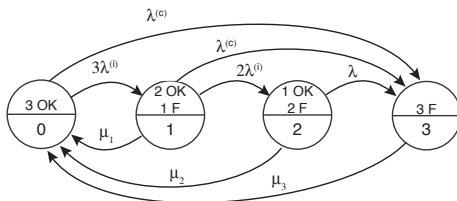
$$P_2 = \frac{6\mu_3\lambda^2}{6\lambda^3 + 11\lambda^2\mu_3 + 5\lambda\mu_2\mu_3 + \mu_1\mu_2\mu_3 + \lambda\mu_1\mu_3} \approx 1.12 \cdot 10^{-6}$$

$$P_3 = \frac{6\lambda^3}{6\lambda^3 + 11\lambda^2\mu_3 + 5\lambda\mu_2\mu_3 + \mu_1\mu_2\mu_3 + \lambda\mu_1\mu_3} \approx 5.99 \cdot 10^{-10}$$

The numerical results assume parameter values $\lambda = 5.0 \cdot 10^{-4}$ per hour (i.e., MTTF = 2000 hours) and $\mu = 10^{-1}$ per hour (i.e., MTTR = 10 hours).

Example: 2oo3 System (cont.)

Now we expand our analysis of the subsystem of three identical channels to also include common cause failures. We have then two types of failures: independent failures (with $\lambda^{(i)}$) as well as CCFs (with $\lambda^{(c)}$). This modification does not introduce new states, but new and modified transitions.



Example: 2003 System (cont.)

The state transition matrix becomes:

$$\mathbb{A} = \begin{pmatrix} -(3\lambda^{(i)} + \lambda^{(c)}) & 3\lambda^{(i)} & 0 & \lambda^{(c)} \\ \mu_1 & -(\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}) & 2\lambda^{(i)} & \lambda^{(c)} \\ \mu_2 & 0 & -(\mu_2 + \lambda) & \lambda \\ \mu_3 & 0 & 0 & -\mu_3 \end{pmatrix}$$

By using the same parameter values as case study I, and knowing that $\lambda^{(i)} = (1 - \beta)\lambda$ and $\lambda^{(c)} = \beta\lambda$ and $\beta = 0.10$, we get:

$$P_0 = \approx 0.9989$$

$$P_1 = \approx 1.08 \cdot 10^{-3}$$

$$P_2 = \approx 9.70 \cdot 10^{-7}$$

$$P_3 = \approx 5.00 \cdot 10^{-5}$$

Example: 2003 System (cont.)

It may be of interest to determine how often (frequent) a state is visited, for example how often it enters a failed state.

With basis in case study II, we find that the visit frequencies are:

$$\nu_0 = P_0 \alpha_0 = P_0(3\lambda^{(i)} + \lambda^c) \approx 1.40 \cdot 10^{-4} \text{ per hour}$$

$$\nu_1 = P_1 \alpha_1 = P_1(\mu_1 + 2\lambda^{(i)} + \lambda^c) \approx 1.35 \cdot 10^{-4} \text{ per hour}$$

$$\nu_2 = P_2 \alpha_2 = P_2(\mu_2 + \lambda^{(i)} + \lambda^c) \approx 9.70 \cdot 10^{-8} \text{ per hour}$$

$$\nu_3 = P_3 \alpha_3 = P_3 \mu_3 \approx 5.00 \cdot 10^{-6} \text{ per hour}$$

Example: 2003 System (cont.)

A2003 system fails when it enters state 2 or state 3. This means that the frequency of system failures, ω_F is:

$$\omega = P_1 2\lambda^{(i)} + (P_0 + P_1)\lambda^c \approx 5.10 \cdot 10^{-5} \text{ per hour}$$

which means that the system will fail on the average once every 22.4 years.

Steps

The main approach to determine MTTF (for first failure) is:

1. Remove transitions from states where the system is in the failed state. These states now defined as **absorbing states**
2. Set up the corresponding *modified* transition matrix (having now the rows corresponding to absorbing states as just “zeros”).
3. Set up the *reduced* transition matrix (A_{red}), where the rows of zeros are removed, along with the columns that represent the absorbing states
4. Solve for MTTF using either approach 1 or approach 2:
 - Approach 1: Setting up and solving the differential equations
 - Approach 2: Setting up and solving the Laplace transform of the differential equations

Step 1-3 for a 2oo3 System

Step 1: Consider again the 2oo3 system with the CCFs included. The original transition matrix was:

$$\mathbb{A} = \begin{pmatrix} -(3\lambda^{(i)} + \lambda^{(c)}) & 3\lambda^{(i)} & 0 & \lambda^{(c)} \\ \mu_1 & -(\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}) & 2\lambda^{(i)} & \lambda^{(c)} \\ \mu_2 & 0 & -(\mu_2 + \lambda) & \lambda \\ \mu_3 & 0 & 0 & -\mu_3 \end{pmatrix}$$

Step 2: This system fail in state 3 and state 4, and we remove these transitions. The modified transition matrix becomes then:

$$\mathbb{A}_{mod} = \begin{pmatrix} -(3\lambda^{(i)} + \lambda^{(c)}) & 3\lambda^{(i)} & 0 & \lambda^{(c)} \\ \mu_1 & -(\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}) & 2\lambda^{(i)} & \lambda^{(c)} \\ 0 & 0 & -(\mu_2 + \lambda) & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 3: The reduced transition matrix consist of the modified matrix, but with row no 4 removed (because of all zero's) and column 3 and 4 removed (because they are absorbing states). This gives:

$$\mathbb{A}_{red} = \begin{pmatrix} -(3\lambda^{(i)} + \lambda^{(c)}) & 3\lambda^{(i)} \\ \mu_1 & -(\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}) \end{pmatrix}$$

Step 4: Approach 1

The approach for solving time-dependent probabilities is as follows:

- ▶ Find survival function $R_s(t)$ using the reduced transition matrix, which is the sum of the state probabilities where the system is in a functioning state.
- ▶ Calculate MTTF using:

$$\text{MTTF} = \int_0^{\infty} R(t) dt$$

This may be done by hand calculation or by using e.g. MAPLE® or MATLAB®.

Step 4: Results

Consider again the subsystem of three identical and independent channels with failure rate λ voted 2oo3. Since the channels are identical, it suffices to define four states.

We define both state 3 and state 4 as absorbing, and remove the transitions from state 3 to state 0 and from state 2 to state 0. The result is that we can set up the following simplified matrix:

$$\mathbb{A}_{red} = \begin{pmatrix} -(3\lambda^{(i)} + \lambda^{(c)}) & 3\lambda^{(i)} \\ \mu_1 & -(\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}) \end{pmatrix}$$

Using data used earlier for this example and solving by e.g. MAPLE, we get $MTTF \approx 1.962 \cdot 10^5$ hours or ≈ 22.4 years.

Step 4: Approach 2

The approach for solving Laplace transforms is based on the following assumption:

$$R(\infty) = R^*(0) = \int_0^{\infty} e^{-t} R(t) dt = \int_0^{\infty} R(t) dt = \text{MTTF}$$

where R^* is calculated for the functioning states.

Reconsider again the 2003 voted system. The reduced transition matrix only contains transitions to and from functioning states. The Laplace transform of the reduced state probabilities becomes:

$$\begin{bmatrix} P_0^*(s), P_1^*(s) \end{bmatrix} \mathbb{A}_{red} = \begin{bmatrix} sP_0^*(s) - P_0(0), sP_1^*(s) - P_1(0) \end{bmatrix}$$

Because the voted group is started in state 0, $P_0(0) = 1$ and $P_1(0) = 0$, and the equation becomes

$$\begin{bmatrix} P_0^*(s), P_1^*(s) \end{bmatrix} \mathbb{A}_{red} = \begin{bmatrix} sP_0^*(s) - 1, sP_1^*(s) \end{bmatrix}$$

Letting $s = 0$ is the same as letting $t \rightarrow \infty$:

$$\begin{bmatrix} P_0^*(0), P_1^*(0) \end{bmatrix} \mathbb{A}_{red} = [-1, 0]$$

Step 4: Results

With the reduced transition matrix for the 2003 voted system with CCFs included, we can now solve the equation. This can be done by hand, or by using e.g. MAPLE. Using MAPLE, we got:

$$P_0^*(0) = \frac{\mu_1 + 2\lambda^{(i)} + \lambda^{(c)}}{6\lambda^{(i)2} + 5\lambda^{(i)}\lambda^{(c)} + \lambda^{(c)}\mu_1 + \lambda^{(c)2}}$$

$$P_1^*(0) = \frac{3\lambda^{(i)}}{6\lambda^{(i)2} + 5\lambda^{(i)}\lambda^{(c)} + \lambda^{(c)}\mu_1 + \lambda^{(c)2}}$$

This means that:

$$\begin{aligned} \text{MTTF} &= R^*(0) = P_0^*(0) + P_1^*(0) \\ &= \frac{3\lambda^{(i)} + \mu_1 + 2\lambda^{(i)} + \lambda^{(c)}}{6\lambda^{(i)2} + 5\lambda^{(i)}\lambda^{(c)} + \lambda^{(c)}\mu_1 + \lambda^{(c)2}} \end{aligned}$$

When using the same data as before:

$$\text{MTTF} \approx 22.40 \text{ years}$$

This is the same result as we got when we solved for the time dependent solution.