Chapter 3
Structure Analysis

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(Version 0.1)

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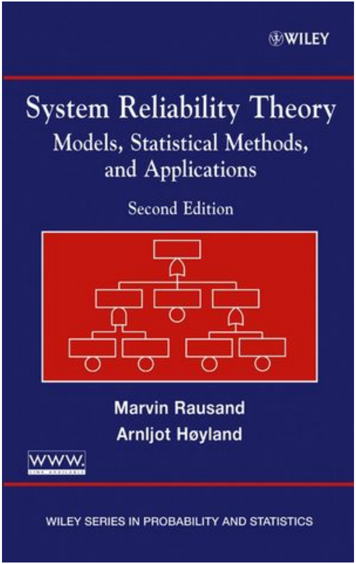
Slides related to the book

System Reliability Theory
Models, Statistical Methods,
and Applications

Wiley, 2004

Homepage of the book:
http://www.ntnu.edu/ross/books/srt



State variable

Consider a system with n individual components. This system is said to have order n. The components are referred to by the numbers $1, 2, \ldots, n$.

Each component can have two possible states: functioning and failed.

The state of component i is given by the binary variable x_i,

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is in a failed state} \end{cases}$$

x_i is called the state variable of component i.

The state vector for the system is

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

Structure function

The state of the system (of order n) can be described by the binary function

$$\phi(\mathbf{x}) = \phi(x_1, x_2, \ldots, x_n)$$

where

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

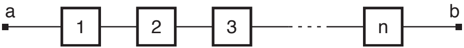
$\phi(\mathbf{x})$ is called the structure function of the system or just the structure.

We often talk about structures of order n instead of systems of order n.

Series structure

A system that is functioning if and only if *all* of its n components are functioning, is called a series structure.

A series structure of order n is illustrated by the reliability block diagram



The structure function of the series structure is

$$\phi(\mathbf{x}) = x_1 \cdot x_2 \cdots x_n = \prod_{i=1}^{n} x_i$$

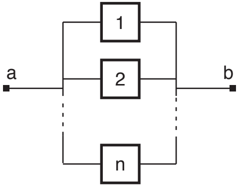
The state, $\phi(\mathbf{x})$, of the series structure is equal to 1 (i.e., functioning) if and only if all the n components have state 1 (i.e., functioning).

The series structure fails (i.e., has state 0) as soon as at least one of its components fails (i.e., has state 0).

Parallel structure - 1

A system that is functioning whenever at least one of its n components is functioning, is called a parallel structure.

A parallel structure of order n is illustrated by the reliability block diagram



The structure function of the parallel structure is

$$\phi(\mathbf{x}) = 1 - \prod_{i=1}^{n}(1 - x_i)$$

The state, $\phi(\mathbf{x})$, of the parallel structure is equal to 1 (i.e., functioning) if at least one of the n components have state 1 (i.e., functioning).

Parallel structure - 2

The structure function of a parallel structure of order $n = 2$ is hence:

$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

Sometimes, we use the symbol $\amalg$ and write

$$\phi(\mathbf{x}) = x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

$x_1 \amalg x_2$ may be read as "component 1 in parallel with component 2."

We may also use the same symbol for a parallel structure of order n

$$\phi(\mathbf{x}) = x_1 \amalg x_2 \amalg \cdots \amalg x_n = \coprod_{i=1}^{n} x_i$$

k-out-of-n structure - 1

A structure that is functioning if and only if at least k of the n components are functioning, is called a k-out-of-n structure

The structure function of a k-out-of-n structure can be written

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if} \quad \sum_{i=1}^{n} x_i < k \end{cases}$$

k-out-of-n structure - 2

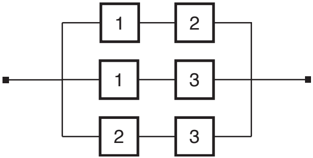
A k-out-of-n structure is often called a $k$oo$n$ structure.

Notice that:

- A 1oo1 structure is a single component
- An $n$oo$n$ structure is a series structure of n components (All the n components must function for the structure to function)
- A 1oo$n$ structure is a parallel structure of n components (It is sufficient that one component is functioning for the structure to function)

2-out-of-3 structure

A 2-out-of-3 (2oo3) structure is functioning when at least 2 of its 3 components are functioning.



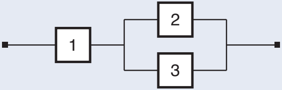
Note that each component appears twice in the reliability block diagram. Note also that since x_i is a binary variable, $x_i^k = x_i$ for all i and k.

$$\begin{aligned}\phi(\mathbf{x}) &= x_1x_2 \amalg x_1x_3 \amalg x_2x_3 \\ &= 1-(1-x_1x_2)(1-x_1x_3)(1-x_2x_3) \\ &= x_1x_2+x_1x_3+x_2x_3-x_1^2x_2x_3-x_1x_2^2x_3-x_1x_2x_3^2+x_1^2x_2^2x_3^2 \\ &= x_1x_2+x_1x_3+x_2x_3-2x_1x_2x_3\end{aligned}$$

Example

Example: Structure function

Consider the structure of order 3:



The structure function is

$$\begin{aligned}\phi(\mathbf{x}) &= x_1 \cdot (x_2 \amalg x_3) \\ &= x_1(x_2 + x_3 - x_2x_3)\end{aligned}$$

When each state variable x_i does not appear such that it will not be multiplied by itself, it is not necessary to perform the multiplication, i.e., the expression can be kept as it is.

Irrelevant component - 1

☞ Irrelevant component: A component is irrelevant for a structure when it does not play any direct role for the functioning ability of the structure, i.e., such that $\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x})$ for all $(\cdot_i, \mathbf{x})$.

Notation:

The following notation is used: $(1_i, \mathbf{x})$ is a conditional state vector where it is known that component i is functioning (i.e., has state 1). This means

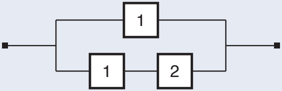
$$(1_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, x_{i+2}, \ldots, x_n)$$

In the same way, $(0_i, \mathbf{x})$ is the state vector when we know that $x_i = 0$, and $(\cdot_i, \mathbf{x})$ is the state vector when the state of component i is not considered.

Irrelevant component - 2

Example: Irrelevant component

Consider the structure of order 2



This structure is functioning when component 1 is functioning, irrespective of whether component 2 is functioning or not. Component 2 is therefore irrelevant in this case.

The structure function is

$$\phi(\mathbf{x}) = x_1$$

Minimal path sets - 1

☞ Path set: A set of components of a structure that by functioning ensures that the structure is functioning.

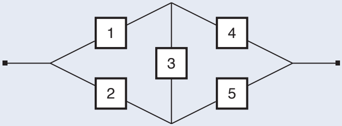
☞ Minimal path set: A path set of a structure that cannot be reduced without loosing status as a path set.

- If all the components of a path set are functioning, the structure is functioning.
- The statement "cannot be reduced" means that: If we remove one component from a minimal path set, the set is no longer a path set.
- The number of components of a minimal path set is called the order of the minimal path set.

Minimal path sets - 2

Example: Minimal path set

Consider the bridge structure



Path sets: $\{1, 4\}, \{2, 5\}, \{1, 3, 5\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 5\}$, etc. The first four of these path sets are minimal such that if we remove one component, the set is no longer a path set. The remaining path sets are seen not to be minimal.

The minimal path sets are therefore

$P_1 = \{1, 4\}, P_2 = \{2, 5\}, P_3 = \{1, 3, 5\}, P_4 = \{2, 3, 4\}$.

The bridge structure has two minimal path sets of order 2 and two minimal path sets of order 3.

Minimal cut sets - 1

☞ Cut set: A set of components of a structure that by failing ensures that the structure is in a failed state.

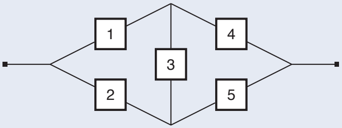
☞ Minimal cut set: A cut set of a structure that cannot be reduced without loosing status as a cut set.

- If all the components of a cut set are failing, the structure is in a failed state.
- The statement “cannot be reduced” means that: If we remove one component from a minimal cut set, the set is no longer a cut set.
- The number of components of a minimal cut set is called the order of the minimal cut set.

Minimal cut sets - 2

Example: Minimal cut set

Again, consider the bridge structure



Cut sets: $\{1, 2\}, \{4, 5\}, \{1, 3, 5\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 4, 5\}$, etc. The first four of these path sets are minimal such that if we remove one component, the set is no longer a cut set. The remaining cut sets are not minimal.

The minimal cut sets are therefore

$C_1 = \{1, 2\}, P_2 = \{4, 5\}, P_3 = \{1, 3, 5\}, P_4 = \{2, 3, 4\}$

The bridge structure has two minimal cut sets of order 2 and two minimal cut sets of order 3.

Minimal path series structure

☞ Minimal path series structure: A series structure of the components of a minimal path set.

The bridge structure has four minimal path series structures:

$$\rho_1(\mathbf{x}) = x_1 x_4$$
$$\rho_2(\mathbf{x}) = x_2 x_3$$
$$\rho_3(\mathbf{x}) = x_1 x_3 x_5$$
$$\rho_4(\mathbf{x}) = x_2 x_3 x_4$$

The bridge structure is functioning if at least one of these minimal path series structures is functioning. The structure function of the bridge structure can therefore be written as

$$\phi(\mathbf{x}) = \rho_1(\mathbf{x}) \amalg \rho_2(\mathbf{x}) \amalg \rho_3(\mathbf{x}) \amalg \rho_4(\mathbf{x})$$

Minimal cut parallel structure

☞ Minimal cut parallel structure: A parallel structure of the components of a minimal cut set.

The bridge structure has four minimal cut parallel structures:

$$\kappa_1(\mathbf{x}) = x_1 \amalg x_2$$
$$\kappa_2(\mathbf{x}) = x_4 \amalg x_3$$
$$\kappa_3(\mathbf{x}) = x_1 \amalg x_3 \amalg x_5$$
$$\kappa_4(\mathbf{x}) = x_2 \amalg x_3 \amalg x_4$$

The bridge structure is failed if at least one of these minimal cut parallel structures is failed. The structure function of the bridge structure can therefore be written as

$$\phi(\mathbf{x}) = \kappa_1(\mathbf{x}) \cdot \kappa_2(\mathbf{x}) \cdot \kappa_3(\mathbf{x}) \cdot \kappa_4(\mathbf{x})$$

Structure function - 1

Represented by minimal paths/cuts

From the bridge structure example it is apparent that:

The structure function can always be represented as either

- A parallel structure of its minimal path series structures

$$\phi(\mathbf{x}) = \coprod_{i=1}^{p} \rho_i(\mathbf{x}) = 1 - \prod_{i=1}^{p} (1 - \rho_i(\mathbf{x}))$$

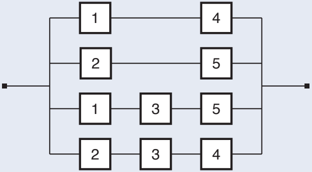
- A series structure of its minimal cut parallel structures

$$\phi(\mathbf{x}) = \prod_{i=1}^{k} \kappa_i(\mathbf{x})$$

Structure function - 2

Example: Bridge structure - minimal paths

The bridge structure can be (equivalently) represented by



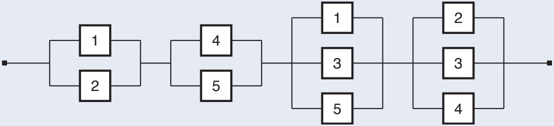
The structure function is

$$
\begin{aligned}
\phi(\mathbf{x}) &= 1 - (1 - \rho_1(\mathbf{x}))(1 - \rho_2(\mathbf{x}))(1 - \rho_3(\mathbf{x}))(1 - \rho_4(\mathbf{x})) \\
&= 1 - (1 - x_1x_4)(1 - x_2x_5)(1 - x_1x_3x_5)(1 - x_2x_3x_4) \\
&= x_1x_4 + x_2x_5 + x_1x_3x_5 + x_2x_3x_4 - x_1x_3x_4x_5 - x_1x_2x_3x_5 \\
&\quad - x_1x_2x_3x_4 - x_2x_3x_4x_5 - x_1x_2x_4x_5 + 2x_1x_2x_3x_4x_5
\end{aligned}
$$

Structure function - 3

Example: Bridge structure - minimal cuts

The bridge structure can be (equivalently) represented by



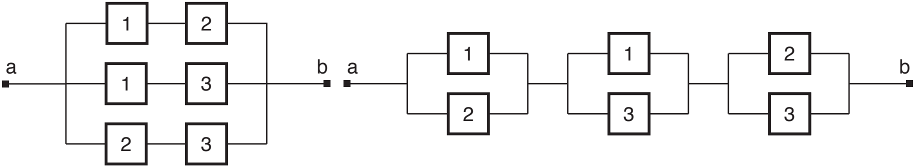
The structure function is

$$\begin{aligned}\phi(\mathbf{x}) &= \kappa_1(\mathbf{x}) \cdot \kappa_2(\mathbf{x}) \cdot \kappa_3(\mathbf{x}) \cdot \kappa_4(\mathbf{x}) \\ &= x_1x_4 + x_2x_5 + x_1x_3x_5 + x_2x_3x_4 - x_1x_3x_4x_5 - x_1x_2x_3x_5 \\ &- x_1x_2x_3x_4 - x_2x_3x_4x_5 - x_1x_2x_4x_5 + 2x_1x_2x_3x_4x_5\end{aligned}$$

which is the same result as obtained from the minimal path sets.

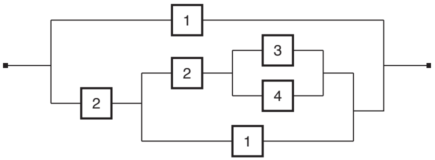
Structure representation

Any coherent structure can be represented as either (i) a parallel structure of its minimal path series structures or as (ii) a series structure of its minimal cut parallel structures – e.g., see the 2oo3 structure

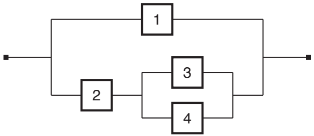


Structure simplification

A coherent structure is determined by its minimal path sets and/or its minimal cut sets. Any reliability block diagram may be redrawn to reflect its minimal path/cut sets. Consider, for example, the structure (Problem 3.7)



A simplified reliability block diagram for this structure is therefore



Path sets of this structure are:
$\{1\}, \{2, 1\}, \{2, 2, 3\}, \{2, 2, 4\}$.

The *minimal* path sets are $\{1\}, \{2, 3\}, \{2, 4\}$

Pivotal decomposition - 1

Pivotal decomposition may also be used to determine the structure function of a structure. It is based on the formula

$$\phi(\mathbf{x}) \equiv x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x}) \quad \text{for all } \mathbf{x}$$

We can easily see that this identity is correct from the fact that

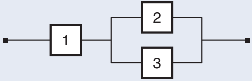
$$x_i = 1 \Rightarrow \phi(\mathbf{x}) = 1 \cdot \phi(1_i, \mathbf{x}) \text{ and } ; x_i = 0 \Rightarrow \phi(\mathbf{x}) = 1 \cdot \phi(0_i, \mathbf{x})$$

By this approach, we determine the structure function by studying two conditional structures – one where component i is known to function, and one where component i is known to have failed.

Pivotal decomposition - 2

Example: Simple structure

Consider the structure of order 3 and choose component $i = 3$ as pivot.



- When component 3 is functioning, component 2 is irrelevant and the structure is reduced to only component 1, i.e., $\phi(1_3, \mathbf{x}) = x_1$
- When component 3 is failed, the structure is reduced to a series structure of components 1 and 2, i.e., $\phi(0_3, \mathbf{x}) = x_1 x_2$

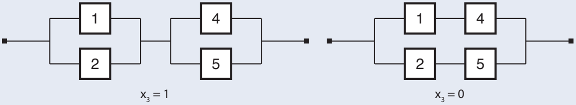
The structure function is therefore

$$\begin{aligned} \phi(\mathbf{x}) &= x_3\phi(1_3, \mathbf{x}) + (1 - x_3)\phi(0_3, \mathbf{x}) \\ &= x_1(x_2 + x_3 - x_2 x_3) \end{aligned}$$

Pivotal decomposition - 3

Example: Bridge structure

We choose $i = 3$ as pivot. The two conditional structures become



The structure functions are easily obtained

$$\phi(1_3, \mathbf{x}) = (x_1 \amalg x_2) \cdot (x_4 \amalg x_5)$$
$$\phi(0_3, \mathbf{x}) = x_1 x_4 \amalg x_2 x_5$$

By entering these into the pivotal decomposition formula, we obtain the same structure function as when using the minimal path/cut sets.

Pivotal decomposition - 4

Pivotal decomposition can be used to find the structure function of any structure.

We can choose the component we like as pivot, but the workload to determine the structure function may vary a lot.

In some cases, we may also need to use pivotal decomposition to determine the structure functions of the conditional substructures $\phi(1_i, \mathbf{x})$ and $\phi(0_i, \mathbf{x})$ – and even for sub- and sub substructures of these.

Determination of the structure function

Four approaches

We have now seen four different approaches to determine the structure function:

1. A direct approach that can be used for purely series-parallel structure – where we proceed from left to right in the reliability block diagram and use the formulas for series and parallel structures.
2. The structure is represented as a parallel structure of the minimal path series structures.
3. The structure is represented as a series structure of the minimal cut parallel structures
4. Pivotal decomposition

All the approaches lead to the same result, but the first approach may not always be feasible (try this approach on the bridge structure)

Critical path vector

A *critical path vector* for component i is a state vector $(1_i, \mathbf{x})$ such that

$$\phi(1_i, \mathbf{x}) = 1 \quad \text{while} \quad \phi(0_i, \mathbf{x}) = 0$$

This is equivalent to requiring that

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1$$

This means that for $(\cdot_i, \mathbf{x})$, the system is functioning if and only if component i is functioning.

We sometimes say that for $(\cdot_i, \mathbf{x})$, component i is *critical* for the system.

Critical component 1

☞ Critical component: Component i is critical for a system when the other system components have such states states $(\cdot_i, \mathbf{x})$ that the system is functioning if and only if component i is functioning.

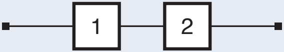
When we say that component i is critical for the system, this is not a statement about the state of component i, but a statement about the states of the other system components.

When a critical component i has state 0, the system is in a failed state. If we repair component i, i.e., bring its state from 0 to state 1, the system will start functioning again.

Critical component 2

Example 1: Series structure

Consider a series structure of two components, 1 and 2.



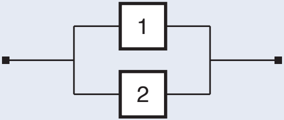
Component 1 can only be critical for the system when component 2 is functioning, i.e., when $x_2 = 1$. In this case, the system is functioning if and only if component 1 is functioning. If component 2 is failed, i.e., $x_2 = 0$, the state of component does not matter.

The critical path vector for component 1 is therefore $(\cdot_1, \mathbf{x}) = (\cdot, 1)$

Critical component 3

Example 2: Parallel structure

Consider a parallel structure of two components, 1 and 2.



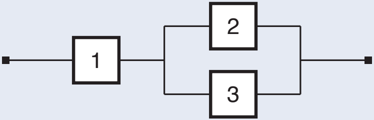
Component 1 can only be critical when component 2 is failed, i.e., when $x_2 = 0$, in which case the system is functioning if and only if component 1 is functioning. When component 2 is functioning, i.e., $x_2 = 1$, the state of component 1 does not matter.

The critical path vector for component 1 is therefore $(\cdot_1, \mathbf{x}) = (\cdot, 0)$

Critical component

Example 3: Series-parallel structure

Consider a structure of three components, 1, 2, and 3.



Component 3 is here critical when component 1 is functioning and component 2 is failed.

The critical path vector for component 3 is therefore $(\cdot_3, \mathbf{x}) = (1, 0, \cdot)$.

System modules

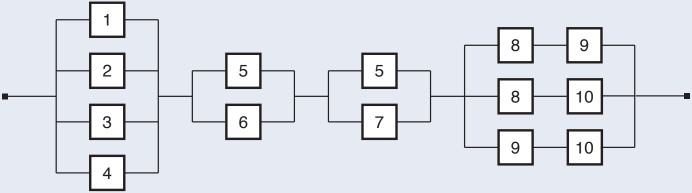
A coherent system can sometimes be partitioned into two or more coherent subsystems, called *modules*.

When the system is partitioned into modules, it is important that *each single component never appears within more than one module*, i.e., that the modules are disjoint.

System modules

Example

Consider the system represented by the reliability block diagram:

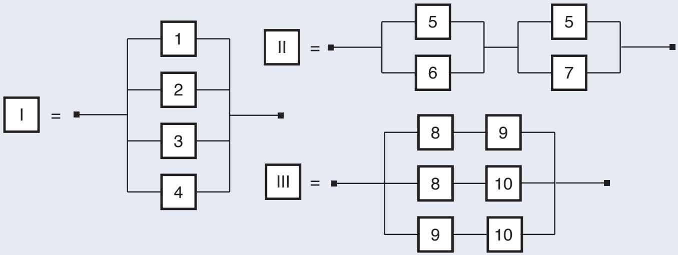


This system can be partitioned into three (or more) modules.

System modules

Example (cont.)

The three modules are defined as follows



Module II cannot be further partitioned because component 5 is present in both parallel structures.