Chapter 9

Optimization of Replacement Intervals

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Slides related to the book

System Reliability Theory
Models, Statistical Methods, and Applications
Wiley, 2004

Homepage of the book:
http://www.ntnu.edu/ross/books/srt
Two different criteria

- Time based replacement
  - Age replacement
  - Block replacement

- Condition based replacement
  - Continuous deterioration
  - Deterioration following a shock
**Age replacement policy**

- Planned replacement, cost = $c$
- Failure, replacement cost = $c + k$
- Planned replacement, cost = $c$

$c = \text{Planned replacement cost}$

$c + k = \text{Unplanned replacement cost}$

$t_0 = \text{Planned replacement age}$
Age replacement - Examples

- Engine drive chain of (some) automobiles
  - On old Volvos the drive chain should be replaced every 80 000 km
- Oil and oil filters in automobiles
- Parts in airplanes
Age replacement

- The mean time between replacements:

\[
\text{MTBR}(t_0) = \int_0^{t_0} tf(t) \, dt + t_0 \cdot \Pr(T \geq t_0)
= \int_0^{t_0} (1 - F(t)) \, dt
\]

- The mean cost per replacement period:

\[
c + k \cdot \Pr(\text{failure}) = c + k \cdot \Pr(T < t_0)
= c + k \cdot F(t_0)
\]
The mean cost per time unit, $C_A(t_0)$, is determined from

$$C_A(t_0) \cdot \text{MTBR}(t_0) = c + k \cdot F(t_0)$$

The cost optimal replacement interval $t_0$ can therefore be found by minimizing

$$C_A(t_0) = \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt}$$

with respect to $t_0$. 
Cost optimization criterion - 2

If we let $t_0 \to \infty$, we get

$$C_A(\infty) = \lim_{t_0 \to \infty} C_A(t_0) = \frac{c + k}{\int_0^\infty (1 - F(t)) \, dt} = \frac{c + k}{\text{MTTF}}$$

which is an obvious result, since no age replacements will take place. Now consider

$$\frac{C_A(t_0)}{C_A(\infty)} = \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt} \cdot \frac{\text{MTTF}}{c + k}$$

$$= \frac{1 + r \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt} \cdot \frac{\text{MTTF}}{1 + r}$$

where $r = k/c$. 
The ratio

\[
\frac{C_A(t_0)}{C_A(\infty)}
\]

may be used as a measure of the cost efficiency of the age replacement policy with replacement interval \( t_0 \). A low value of the ratio \( C_A(t_0)/C_A(\infty) \) indicates a high cost efficiency.
Example 9.9 -1

Let $T \sim \text{Weibull} \ (\alpha, \lambda)$ where $\alpha > 1$. We get

$$\frac{C_A(t_0)}{C_A(\infty)} = 1 + r \left(1 - e^{-(\lambda t_0)^\alpha}\right) \cdot \frac{\Gamma(1/\alpha + 1)/\lambda}{\int_0^{t_0} e^{-(\lambda t)^\alpha} \, dt} \cdot \frac{1}{1 + r}$$

By introducing $x_0 = \lambda t_0$, we get

$$\frac{\tilde{C}_A(x_0)}{C_A(\infty)} = 1 + r \left(1 - e^{-x_0^\alpha}\right) \cdot \frac{\Gamma(1/\alpha + 1)}{\int_0^{x_0} e^{-x^\alpha} \, dx} \cdot \frac{1}{1 + r}$$

We have to use a computer to find the $x_0$ that minimizes $\frac{\tilde{C}_A(x_0)}{C_A(\infty)}$. 
Example 9.9 - 2

The ratio $\frac{\tilde{C}_A(x_0)}{C_A(\infty)}$ as a function of $x_0$ for the Weibull distribution with shape parameter $\alpha = 3$, and $r = 3, 5, \text{ and } 10$. 

![Graph showing the ratio of $\tilde{C}_A(x_0)/C_A(\infty)$ as a function of $x_0$ for different values of $r$.]
Availability criterion - 1

In some applications we may be interested in finding the age replacement interval \( t_0 \) that minimizes the *average unavailability* of the item. Mean downtime in a replacement period

\[
\text{MDT}(t_0) = \text{MDT}_F \cdot F(t_0) + \text{MDT}_P \cdot (1 - F(t_0))
\]

where:

- \( \text{MDT}_F \) = the mean downtime due to a failure
- \( \text{MDT}_P \) = the mean downtime due to a preventive replacement
The mean time between replacements is

\[ \text{MTBR}(t_0) = \int_0^{t_0} (1 - F(t)) \, dt \]

\[ + \text{MDT}_F \cdot F(t_0) + \text{MDT}_P \cdot (1 - F(t_0)) \]

The optimal \( t_0 \) then is the \( t_0 \) that minimizes

\[ \bar{A}_{av}(t_0) = \frac{\text{MDT}(t_0)}{\text{MTBR}(t_0)} \]
Age replacement problems

- Has to assume that the item is replaced with an “as good as new” item (not worse and not better)
- Has to monitor the age of the item (may be difficult to administer when we have many items)
- Maintenance actions will be spread out in time
The item is replaced at regular time intervals $t_0, 2t_0, \ldots$ regardless of age.

If an item fails within an interval, it is repaired (minimal, imperfect, or perfect).

Let $N(t_0)$ be the number of failures/repairs within an interval of length $t_0$.

Let $W(t_0) = E(N(t_0))$.

The cost optimal replacement interval is found by minimizing

$$C_B(t_0) = \frac{c + k \cdot W(t_0)}{t_0}$$

with respect to $t_0$. 
Block replacement - Example

- Replace all light bulbs in a building every 1. January
- Better examples??
Example 9.10 - 1

Assume that the replacement interval is so short that the probability of 2 or more replacements within \((0, t_0)\) is negligible. In this case

\[ W(t_0) = E(N(t_0)) \approx \Pr(N(t_0) = 1) = F(t_0) \]

The average cost \(C_B(t_0)\) per time unit is

\[ C_B(t_0) \approx \frac{c + k \cdot F(t_0)}{t_0} \]

The minimum of \(C_B(t_0)\) may be found by solving \(dC_B(t_0)/dt_0 = 0\), which gives

\[ \frac{c}{k} + F(t_0) = t \cdot F'(t_0) \]
Let $t \sim \text{Weibull} (\alpha, \lambda)$ where $\alpha > 1$. In this case we get

$$\frac{c}{k} + 1 - e^{(\lambda t_0)^\alpha} = t_0 \cdot \alpha \lambda t_0^{\alpha-1} e^{-(\lambda t_0)^\alpha}$$

which can be written as

$$\frac{c}{k} + 1 = (1 + \alpha (\lambda t_0)^\alpha) \cdot e^{-(\lambda t_0)^\alpha}$$

For this model to be realistic the preventive replacement cost $c$ must be small compared to the corrective replacement cost $k$. 
Example 9.10 - 3

The optimal replacement interval $t_0$ as a function of the shape parameter $\alpha$ of the Weibull distribution. The optimal value $t_0$ is equal to $h \cdot \text{MTTF}.$
Summary

- The block replacement policy is easier to administer than the age replacement policy
- We may have to replace rather new items
- Different types of “intermediate” repair may be considered (minimal, imperfect, perfect)
- Problems related to number of spares available may be considered
- The policy may be optimized with respect to cost and/or availability
Condition-based - 1

- $Y(t) = \text{the deterioration of the item as a function of time } t$
- The deterioration is modeled as a stochastic process; usually as a \textit{gamma process}, or a Wiener process
Condition-based - 2

- The item is inspected, and the deterioration “measured”, at times $t_1, t_2, \ldots$
- When a measurement is $\geq y_p$ the item is preventively replaced at cost $c$
- When a measurement is $\geq y_c$ the item is correctively replaced at cost $c + k$
Example 9.14 - 1

- Deterioration $Y(t)$ is modeled as a gamma process
- Inspection interval: $\tau$
- Deterioration in intervals $\Delta Y_1, \Delta Y_2, \ldots$ are independent and gamma distributed with distribution function $F(y)$, and mean $\mu \tau$
- let $n_p =$ mean number of inspections until the deterioration reaches the threshold $y_p$, that is $n_p \cdot \mu \tau \approx y_p$
- The crossing will be detected in inspection $\tilde{n}_p$ where

$$\tilde{n}_p = \frac{y_p}{\mu \tau} + 1$$
Example 9.14 - 2

- The mean time between replacements is

\[ MTBR(\tau) = \left( \frac{y_p}{\mu \tau} + 1 \right) \cdot \tau \]

- The average cost per replacement cycle is

\[
\begin{align*}
  c + k_i \cdot n_p + k \cdot \Pr(\text{failure}) \\
  = c + k_i \cdot n_p + k \cdot \Pr(\Delta Y > (y_c - y_p)) \\
  = c + k_i \cdot n_p + k \cdot \left[ 1 - F(y_c - y_p) \right]
\end{align*}
\]
In some applications it has been found to be realistic to model the deterioration as a gamma stochastic process \( \{Y(t), t \geq 0\} \), with the following characteristics:

1. \( Y(0) = 0 \).
2. The process \( \{Y(t), t \geq 0\} \) has independent increments.
3. For all \( 0 \leq s < t \), the random variable \( Y(t) - Y(s) \) has a gamma distribution with parameters \( (\alpha(t - s), \beta) \), with probability density function

\[
f_{(s,t)}(y) = \frac{\beta}{\Gamma(\alpha(t - s))} (\beta y)^{\alpha(t-s)-1} e^{-\beta y} \quad \text{for } y \geq 0
\]
The mean deterioration in the interval \((s, t)\) is:

\[
E[Y(t) - Y(s)] = \frac{\alpha}{\beta} \ (t - s)
\]

When using the gamma process, the mean deterioration is therefore a linear function of time with deterioration speed (slope) \(\alpha/\beta\).
The item is inspected after regular intervals of length $\tau$

Shock occurs according to a homogeneous Poisson process with rate $\lambda$

A potential failure (following a shock) is detectable at time $P$

The item is functionally failed at time $F$
\( C_P = \) the cost of a preventive replacement

\( C_C = \) the cost of a corrective replacement

\( C_I = \) the cost of an inspection

Objective:

To find the test interval that minimizes the total cost!

The following situations are treated in the book:

- Deterministic PF interval and repair time, perfect inspection
- Stochastic PF interval, deterministic repair time, and non-perfect inspection
Example 9.16

- Cracks in (railway) rails
- Frequency of initiated cracks depends on traffic load, rail material, rail geometry, particles on the rails, shocks from trains with non-circular wheels, and so on
- Inspection by a special rail-car with ultrasonic inspection equipment (implies high cost)
- Detection probability depends on the depth of the crack
- PF interval is the time interval from the crack is observable until a critical failure occurs
- A critical failure may involve derailment and fatalities (i.e., $C_C$ difficult to assess)