Chapter 9
Optimization of Replacement Intervals

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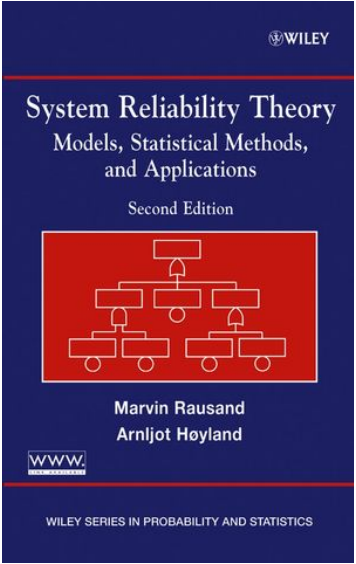
Slides related to the book

System Reliability Theory
Models, Statistical Methods,
and Applications

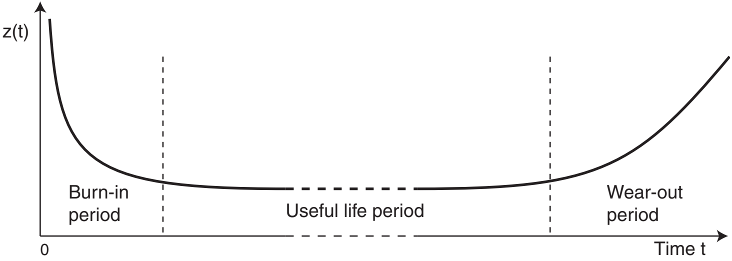
Wiley, 2004

Homepage of the book:
http://www.ntnu.edu/ross/books/srt

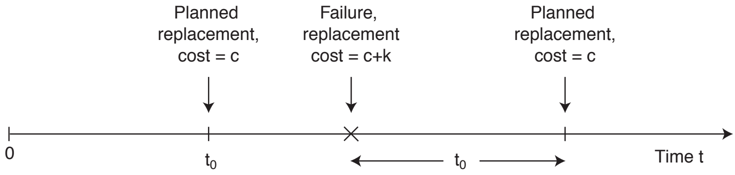


Two different criteria

- Time based replacement
  - Age replacement
  - Block replacement
- Condition based replacement
  - Continuous deterioration
  - Deterioration following a shock



Age replacement policy



c	=	Planned replacement cost
$c + k$	=	Unplanned replacement cost
t_0	=	Planned replacement age

Age replacement - Examples

- Engine drive chain of (some) automobiles
  - On old Volvos the drive chain should be replaced every 80 000 km
- Oil and oil filters in automobiles
- Parts in airplanes

Age replacement

- The mean time between replacements:

$$\begin{aligned}\text{MTBR}(t_0) &= \int_0^{t_0} tf(t)\,dt + t_0 \cdot \Pr(T \geq t_0) \\ &= \int_0^{t_0} (1 - F(t))\,dt\end{aligned}$$

- The mean cost per replacement period:

$$\begin{aligned}c + k \cdot \Pr(\text{failure}) &= c + k \cdot \Pr(T < t_0) \\ &= c + k \cdot F(t_0)\end{aligned}$$

Cost optimization criterion - 1

The mean cost per time unit, $C_A(t_0)$, is determined from

$$C_A(t_0) \cdot \mathrm{MTBR}(t_0) = c + k \cdot F(t_0)$$

The cost optimal replacement interval t_0 can therefore be found by minimizing

$$C_A(t_0) = \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t))\, dt}$$

with respect to t_0.

Cost optimization criterion - 2

If we let $t_0 \to \infty$, we get

$$C_A(\infty) = \lim_{t_0 \to \infty} C_A(t_0) = \frac{c + k}{\int_0^\infty (1 - F(t))\, dt} = \frac{c + k}{\text{MTTF}}$$

which is an obvious result, since no age replacements will take place. Now consider

$$\begin{aligned} \frac{C_A(t_0)}{C_A(\infty)} &= \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t))\, dt} \cdot \frac{\text{MTTF}}{c + k} \\ &= \frac{1 + r \cdot F(t_0)}{\int_0^{t_0} (1 - F(t))\, dt} \cdot \frac{\text{MTTF}}{1 + r} \end{aligned}$$

where $r = k/c$.

Cost optimization criterion - 3

The ratio

$$\frac{C_A(t_0)}{C_A(\infty)}$$

may be used as a measure of the *cost efficiency* of the age replacement policy with replacement interval t_0. A low value of the ratio $C_A(t_0)/C_A(\infty)$ indicates a high cost efficiency.

Example 9.9 -1

Let $T \sim$ Weibull (α, λ) where $\alpha > 1$. We get

$$\frac{C_A(t_0)}{C_A(\infty)} = \frac{1 + r\left(1 - e^{-(\lambda t_0)^\alpha}\right)}{\int_0^{t_0} e^{-(\lambda t)^\alpha}\,dt} \cdot \frac{\Gamma(1/\alpha + 1)/\lambda}{1 + r}$$

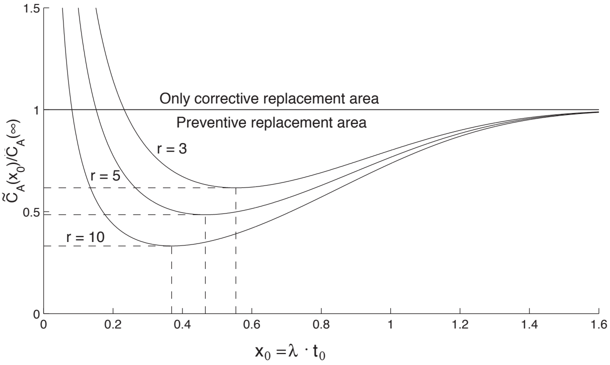
By introducing $x_0 = \lambda t_0$, we get

$$\frac{\tilde{C}_A(x_0)}{C_A(\infty)} = \frac{1 + r\left(1 - e^{-x_0^\alpha}\right)}{\int_0^{x_0} e^{-x^\alpha}\,dx} \cdot \frac{\Gamma(1/\alpha + 1)}{1 + r}$$

We have to use a computer to find the x_0 that minimizes $\tilde{C}_A(x_0)/C_A(\infty)$

Example 9.9 - 2

The ratio $\tilde{C}_A(x_0)/C_A(\infty)$ as a function of x_0 for the Weibull distribution with shape parameter $\alpha = 3$, and $r = 3, 5$, and 10.



Availability criterion - 1

In some applications we may be interested in finding the age replacement interval t_0 that minimizes the *average unavailability* of the item.
Mean downtime in a replacement period

$$\text{MDT}(t_0) = \text{MDT}_F \cdot F(t_0) + \text{MDT}_P \cdot (1 - F(t_0)))$$

where:

- MDT_F = the mean downtime due to a failure
- MDT_P = the mean downtime due to a preventive replacement

Availability criterion - 2

The mean time between replacements is

$$\text{MTBR}(t_0) = \int_0^{t_0} (1 - F(t))\, dt + \text{MDT}_F \cdot F(t_0) + \text{MDT}_P \cdot (1 - F(t_0)))$$

The optimal t_0 then is the t_0 that minimizes

$$\bar{A}_{\text{av}}(t_0) = \frac{\text{MDT}(t_0)}{\text{MTBR}(t_0)}$$

Age replacement problems

- Has to assume that the item is replaced with an “as good as new” item (not worse and not better)
- Has to monitor the age of the item (may be difficult to administer when we have many items)
- Maintenance actions will be spread out in time

Block replacement

- The item is replaced at regular time intervals $t_0, 2t_0, \ldots$ regardless of age
- If an item fails within an interval, it is repaired (minimal, imperfect, or perfect)
- Let $N(t_0)$ be the number of failures/repairs within an interval of length t_0
- Let $W(t_0) = E(N(t_0))$
- The cost optimal replacement interval is found by minimizing

$$C_B(t_0) = \frac{c + k \cdot W(t_0)}{t_0}$$

with respect to t_0

Block replacement - Example

- Replace all light bulbs in a building every 1. January
- Better examples??

Example 9.10 - 1

Assume that the replacement interval is so short that the probability of 2 or more replacements within $(0, t_0)$ is negligible. In this case

$$W(t_0) = E(N(t_0)) \approx \Pr(N(t_0) = 1) = F(t_0)$$

The average cost $C_B(t_0)$ per time unit is

$$C_B(t_0) \approx \frac{c + k \cdot F(t_0)}{t_0}$$

The minimum of $C_B(t_0)$ may be found by solving $dC_B(t_0)/dt_0 = 0$, which gives

$$\frac{c}{k} + F(t_0) = t \cdot F'(t_0)$$

Example 9.10 - 2

Let $t \sim$ Weibull (α, λ) where $\alpha > 1$. In this case we get

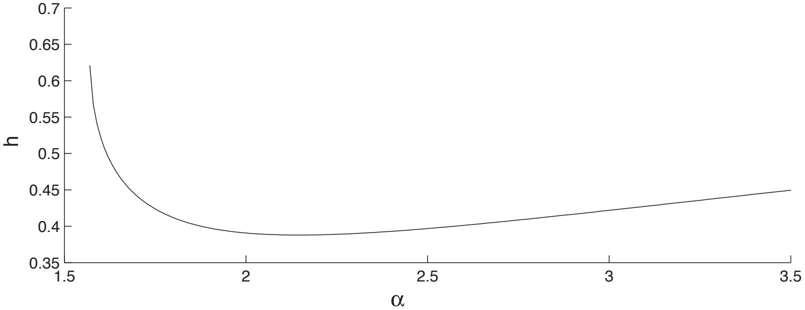
$$\frac{c}{k} + 1 - e^{(\lambda t_0)^\alpha} = t_0 \cdot \alpha \lambda^\alpha t_0^{\alpha-1} e^{-(\lambda t_0)^\alpha}$$

which can be written as

$$\frac{c}{k} + 1 = (1 + \alpha(\lambda t_0)^\alpha) \cdot e^{-(\lambda t_0)^\alpha}$$

For this model to be realistic the preventive replacement cost c must be small compared to the corrective replacement cost k.

Example 9.10 - 3

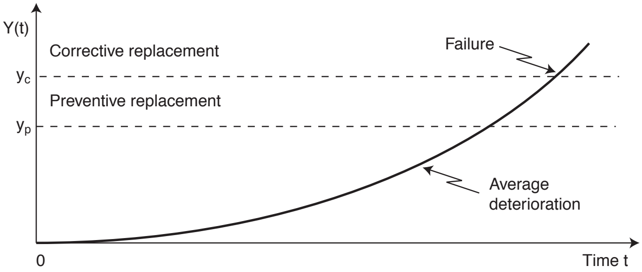


The optimal replacement interval t_0 as a function of the shape parameter α of the Weibull distribution. The optimal value t_0 is equal to $h \cdot \text{MTTF}$.

Summary

- The block replacement policy is easier to administer than the age replacement policy
- We may have to replace rather new items
- Different types of "intermediate" repair may be considered (minimal, imperfect, perfect)
- Problems related to number of spares available may be considered
- The policy may be optimized with respect to cost and/or availability

Condition-based - 1



- $Y(t)$ = the deterioration of the item as a function of time t
- The deterioration is modeled as a stochastic process; usually as a *gamma process*, or a Wiener process

Condition-based - 2

- The item is inspected, and the deterioration "measured", at times $t_1, t_2, \ldots$
- When a measurement is $\geq y_p$ the item is preventively replaced at cost c
- When a measurement is $\geq y_c$ the item is correctively replaced at cost $c + k$

Example 9.14 - 1

- Deterioration $Y(t)$ is modeled as a gamma process
- Inspection interval: τ
- Deterioration in intervals $\Delta Y_1, \Delta Y_2, \ldots$ are independent and gamma distributed with distribution function $F(y)$, and mean $\mu\tau$
- let n_p = mean number of inspections until the deterioration reaches the threshold y_p, that is $n_p \cdot \mu\tau \approx y_p$
- The crossing will be detected in inspection $\tilde{n}_p$ where

$$\tilde{n}_p = \frac{y_p}{\mu\tau} + 1$$

Example 9.14 - 2

- The mean time between replacements is

$$\text{MTBR}(\tau) = \left(\frac{y_p}{\mu\tau} + 1\right) \cdot \tau$$

- The average cost per replacement cycle is

$$\begin{aligned} c + k_i \cdot n_p + k \cdot \Pr(\text{failure}) \\ &= c + k_i \cdot n_p + k \cdot \Pr(\Delta Y > (y_c - y_p)) \\ &= c + k_i \cdot n_p + k \cdot \left[1 - F(y_c - y_p)\right] \end{aligned}$$

Gamma process - 1

In some applications it has been found to be realistic to model the deterioration as a *gamma stochastic process* $\{Y(t), t \geq 0\}$, with the following characteristics:

1. $Y(0) = 0$.
2. The process $\{Y(t), t \geq 0\}$ has independent increments
3. For all $0 \leq s < t$, the random variable $Y(t) - Y(s)$ has a gamma distribution with parameters $(\alpha(t-s), \beta)$, with probability density function

$$f_{(s,t)}(y) = \frac{\beta}{\Gamma(\alpha(t-s))} \, (\beta y)^{\alpha(t-s)-1} \, e^{-\beta y} \quad \text{for } y \geq 0$$

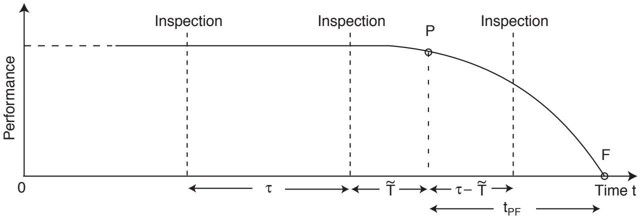
Gamma process - 2

The mean deterioration in the interval (s, t) is:

$$E\left[Y(t) - Y(s)\right] = \frac{\alpha}{\beta}\,(t - s)$$

When using the gamma process, the mean deterioration is therefore a linear function of time with deterioration speed (slope) α/β.

PF intervals - 1



- The item is inspected after regular intervals of length τ
- Shock occur according to a homogeneous Poisson process with rate λ
- A potential failure (following a shock) is detectable at time P
- The item is functionally failed at time F

PF intervals - 2

- C_P = the cost of a preventive replacement
- C_C = the cost of a corrective replacement
- C_I = the cost of an inspection
- Objective:

  To find the test interval that minimizes the total cost!
- The following situations are treated in the book:
  - Deterministic PF interval and repair time, perfect inspection
  - Stochastic PF interval, deterministic repair time, and non-perfect inspection

Example 9.16

- Cracks in (railway) rails
- Frequency of initiated cracks depends on traffic load, rail material, rail geometry, particles on the rails, shocks from trains with non-circular wheels, and so on
- Inspection by a special rail-car with ultrasonic inspection equipment (implies high cost)
- Detection probability depends on the depth of the crack
- PF interval is the time interval from the crack is observable until a critical failure occurs
- A critical failure may involve derailment and fatalities (i.e., C_C difficult to assess)