Chapter 2
Failure Models – Part 1

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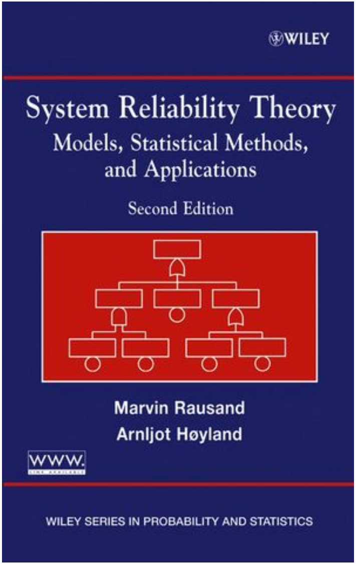
Slides related to the book

System Reliability Theory
Models, Statistical Methods,
and Applications

Wiley, 2004

Homepage of the book:
http://www.ntnu.edu/ross/books/srt



Objectives

This presentation will introduce:

- Reliability measures for non-repaired items:
  - The reliability (survivor) function $R(t)$
  - The failure rate function $z(t)$
  - The mean time to failure (MTTF)
  - The mean residual life (MRL)
- Some common discrete distributions:
  - The binomial distribution
  - The geometric distribution
  - The Poisson distribution – and the Poisson process
- Some common life distributions:
  - The Exponential distribution
  - The Weibull distribution

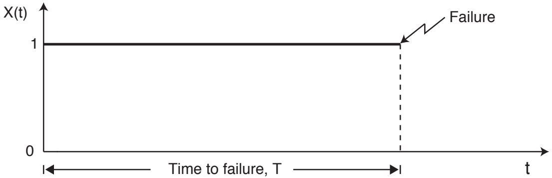
Reliability measures

Here, we only consider an item that is not repaired. The item may be repairable, but we only consider it until the first failure.

Before introducing the reliability measures, we have to define the two concepts:

- State variable
- Time to failure

State variable



$$X(t) = \begin{cases} 1 & \text{if the item can perform a specified function at time } t \\ 0 & \text{if the item cannot perform this function at time } t \end{cases}$$

The state variable $X(t)$ is related to one or more specified functions.

$X(t) = 0$ does not mean that the item is totally failed. It means that the item is not able to perform the specified function(s).

The state variable $X(t)$ and the time to failure T will generally be random variables.

Item

We use the term item to denote any physical component, assembly, or system.

In some applications, the item may be a small component, while it in other applications may be a large system.

Time to failure

The time to failure can be measured by different time concepts

- Calendar time
- Operational time
- Number of kilometers driven by a car
- Number of cycles for a periodically working item
- Number of times a switch is operated
- Number of rotations of a bearing

In most applications, we assume that the time to failure T is a continuous random variable (Discrete variables may be approximated by a continuous variable)

Probability distribution function

The probability distribution function of T is

$$F(t) = \Pr(T \leq t) = \int_0^t f(u)\, du \;\text{ for } t > 0$$

Note that:

- $F(t)$ = The probability that the item will fail within the interval $(0, t]$
- $F(t)$ is sometimes called the failure function of the item
- $F(t)$ is determined by the area under $f(u)$ to the left of t
- $\Pr(t_1 < T \leq t_2) = \Pr(T \leq t_2) - \Pr(T \leq t_1) = F(t_2) - F(t_1)$, i.e., the probability that the item will fail in the interval $(t_1, t_2]$

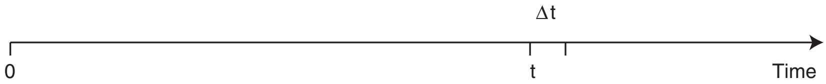
Probability density function

The probability density function (pdf) of T is

$$f(t) = \frac{d}{dt}F(t) = \lim_{\Delta t \to \infty} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to \infty} \frac{\Pr(t < T \leq t + \Delta t)}{\Delta t}$$

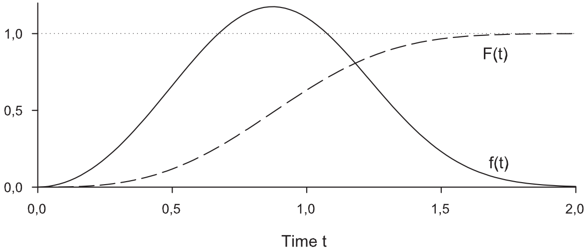
When Δt is small, then

$$\boxed{\Pr(t < T \leq t + \Delta t) \approx f(t) \cdot \Delta t}$$



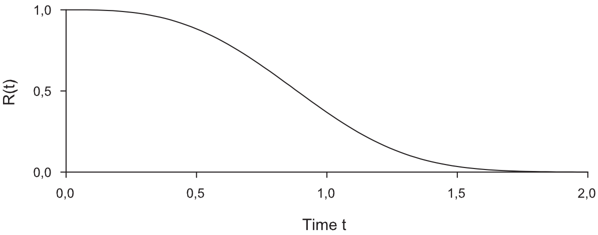
When we are standing at time $t = 0$ and ask: What is the probability that the item will fail in the interval $(t, t + \Delta t]$? The answer is approximately $f(t) \cdot \Delta t$

Distribution of T



- The area under the pdf-curve $f(t)$ is always 1, $\int_0^\infty f(t)\,dt = 1$
- The area under the pdf-curve to the left of t is equal to $F(t)$
- The area under the pdf-curve between t_1 and t_2 is $F(t_2) - F(t_1) = \Pr(t_1 < T \leq t_2)$

Survivor function



$$R(t) = \Pr(T > t) = 1 - F(t) = \int_t^\infty f(u)\,du$$

- $R(t)$ = The probability that the item will not fail in $(0, t]$
- $R(t)$ = The probability that the item will survive at least to time t
- $R(t)$ is also called the reliability function of the item

Failure rate function – 1

Consider the conditional probability

$$\Pr(t < T \le t + \Delta t \mid T > t) = \frac{\Pr(t < T \le t + \Delta t)}{\Pr(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)}$$

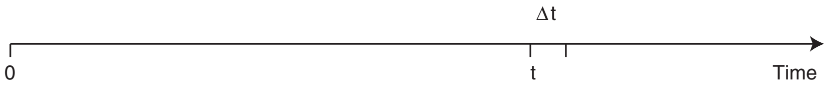
The failure rate function of the item is

$$z(t) = \lim_{\Delta t \to 0} \frac{\Pr(t < T \le t + \Delta t \mid T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)}$$

When Δt is small, we have

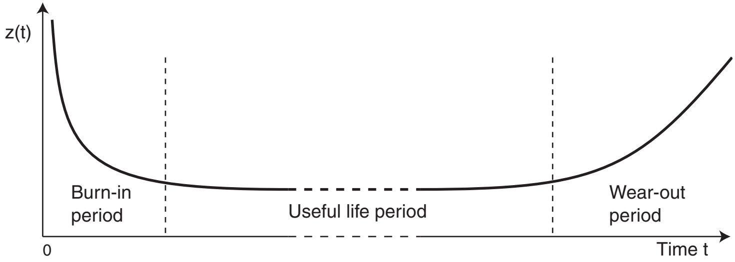
$$\boxed{\Pr(t < T \le t + \Delta t \mid T > t) \approx z(t) \cdot \Delta t}$$

Failure Rate Function – 2



- Note the difference between the failure rate function $z(t)$ and the probability density function $f(t)$.
- When we follow an item from time 0 and note that it is still functioning at time t, the probability that the item will fail during a short interval of length Δt after time t is $z(t)\Delta t$
- The failure rate function is a "property" of the item and is sometimes called the force of mortality (FOM) of the item.

Bathtub curve



Mechanical components are often assumed to have a failure rate function with a bathtub shape.

Some formulas

Expressed by	$F(t)$	$f(t)$	$R(t)$	$z(t)$
$F(t) =$	–	$\int_0^t f(u)\,du$	$1 - R(t)$	$1 - \exp\left(-\int_0^t z(u)\,du\right)$
$f(t) =$	$\frac{d}{dt}F(t)$	–	$-\frac{d}{dt}R(t)$	$z(t) \cdot \exp\left(-\int_0^t z(u)\,du\right)$
$R(t) =$	$1 - F(t)$	$\int_t^\infty f(u)\,du$	–	$\exp\left(-\int_0^t z(u)\,du\right)$
$z(t) =$	$\frac{dF(t)/dt}{1 - F(t)}$	$\frac{f(t)}{\int_t^\infty f(u)\,du}$	$-\frac{d}{dt}\ln R(t)$	–

Mean time to failure

The mean time to failure, MTTF, of an item is

$$\text{MTTF} = E(T) = \int_0^\infty t f(t)\, dt \tag{1}$$

Since $f(t) = -R'(t)$,

$$\text{MTTF} = -\int_0^\infty t R'(t)\, dt$$

By partial integration

$$\text{MTTF} = -\left[tR(t)\right]_0^\infty + \int_0^\infty R(t)\, dt$$

If MTTF $< \infty$, it can be shown that $\left[tR(t)\right]_0^\infty = 0$. In that case

$$\boxed{\text{MTTF} = \int_0^\infty R(t)\, dt} \tag{2}$$

It is often easier to determine MTTF by (2) than by (1).

Example 2.1

Consider an item with survivor function

$$R(t) = \frac{1}{(0.2\,t+1)^2} \quad \text{for } t \geq 0$$

where the time t is measured in months. The probability density function is

$$f(t) = -R'(t) = \frac{0.4}{(0.2\,t+1)^3}$$

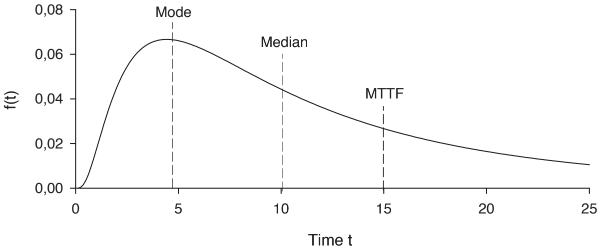
and the failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{0.4}{0.2\,t+1}$$

The mean time to failure is:

$$\text{MTTF} = \int_0^\infty R(t)\,dt = 5 \text{ months}$$

Median



The median life t_m is defined by

$$R(t_m) = 0.50$$

The median divides the distribution in two halves. The item will fail before time t_m with 50% probability, and will fail after time t_m with 50% probability.

Mode

The mode of a life distribution is the most likely failure time, that is, the time t_{mode} where the probability density function $f(t)$ attains its maximum (see figure on the previous slide).

Conditional survivor function

Consider an item that is put into operation at time $t = 0$ and is still functioning at time t. The probability that the item of age t survives an additional interval of length x is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{\Pr(T > x + t)}{\Pr(T > t)} = \frac{R(x + t)}{R(t)}$$

$R(x \mid t)$ is called the conditional survivor function of the item at age t.

With this notation

$$R(x \mid 0) = R(x)$$

Mean residual life

The mean residual life, $\text{MRL}(t)$, of the item at age t is

$$\text{MRL}(t) = \mu(t) = \int_0^\infty R(x \mid t)\, dx = \frac{1}{R(t)} \int_t^\infty R(x)\, dx$$

- MRL is also called the *mean remaining life*
- $\text{MRL}(0)$ of a new item may, for example, be 15 000 hours. When the item is still functioning after 10 000 hours, its $\text{MRL}(10\,000 \text{ hours})$ may, for example, be 8 000 hours. Note that generally $\text{MRL}(x) \neq \text{MRL}(0) - x$
- When we determine $\text{MRL}(x)$ we know (or assume) that the item is functioning at time x, but we do not have any additional information about what happened in the interval $(0, x)$

Example 2.2 – 1

Consider an item with failure rate function $z(t) = t/(t+1)$. The failure rate function is increasing and approaches 1 when $t \to \infty$. The corresponding survivor function is

$$R(t) = \exp\left(-\int_0^t \frac{u}{u+1}\,du\right) = (t+1)\,e^{-t}$$

$$\text{MTTF} = \int_0^\infty (t+1)\,e^{-t}\,dt = 2$$

The conditional survival function is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{(t+x+1)\,e^{-(t+x)}}{(t+1)\,e^{-t}} = \frac{t+x+1}{t+1}\,e^{-x}$$

Example 2.2 – 2

The mean residual life is

$$\text{MRL}(t) = \int_0^\infty R(x \mid t)\, dx = 1 + \frac{1}{t+1}$$

We see that $\text{MRL}(t)$ is equal to 2 (= MTTF) when $t = 0$, that $\text{MRL}(t)$ is a decreasing function in t, and that $\text{MRL}(t) \to 1$ when $t \to \infty$.

Discrete distributions

Three discrete distributions are introduced:

- The binomial distribution
- The geometric distribution
- The Poisson distribution – incl. the homogeneous Poisson process

Binomial distribution – 1

The binomial situation is defined by:

1. We have n independent trials
2. Each trial has two possible outcomes A and A^*
3. The probability $\Pr(A) = p$ is the same in all the n trials

The trials in this situation are sometimes called Bernoulli trials. Let X denote the number of the n trials that have outcome A. The distribution of X is

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad \text{for } x = 0, 1, \ldots, n$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the binomial coefficient.

The distribution is called the binomial distribution (n, p), and we sometimes write $X \sim \text{bin}(n, p)$.

Binomial distribution – 2

The mean value and the variance of X are

$$E(X) = np$$
$$\mathrm{var}(X) = np(1-p)$$

Geometric distribution – 1

Assume that we carry out a sequence of Bernoulli trials, and want to find the number Z of trials until the first trial with outcome A. If $Z = z$, this means that the first $(z-1)$ trials have outcome A^*, and that the first A will occur in trial z. The distribution of Z is

$$\Pr(Z = z) = (1-p)^{z-1} p \quad \text{for } z = 1, 2, \ldots$$

This distribution is called the geometric distribution. We have that

$$\Pr(Z > z) = (1-p)^z$$

Geometric distribution – 2

The mean value and the variance of Z are

$$E(Z) = \frac{1}{p}$$
$$\mathrm{var}(X) = \frac{1-p}{p^2}$$

The geometric distribution is also called the Pascal distribution or the negative binomial distribution.

The homogeneous Poisson process –1

Consider occurrences of a specific event $\mathcal{A}$, and assume that

1. The event $\mathcal{A}$ may occur at any time in the interval, and the probability of $\mathcal{A}$ occurring in the interval $(t, t + \Delta t]$ is independent of t and may be written as $\lambda \cdot \Delta t + o(\Delta t)$, where λ is a positive constant.
2. The probability of more that one event $\mathcal{A}$ in the interval $(t, t + \Delta t]$ is $o(\Delta t)$.
3. Let $(t_{11}, t_{12}], (t_{21}, t_{22}], \ldots$ be any sequence of disjoint intervals in the time period in question. Then the events "$\mathcal{A}$ occurs in $(t_{j1}, t_{j2}]$," $j = 1, 2, \ldots$, are independent.

Without loss of generality we let $t = 0$ be the starting point of the process.

The homogeneous Poisson process – 2

Let $N(t)$ denote the number of times the event $\mathcal{A}$ occurs during the interval $(0, t]$. The stochastic process $\{N(t), t \geq 0\}$ is called a Homogeneous Poisson Process (HPP) with rate λ.

The distribution of $N(t)$ is

$$\Pr(N(t) = n) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots$$

The mean and the variance of $N(t)$ are

$$E(N(t)) = \sum_{n=0}^{\infty} n \cdot \Pr(N(t) = n) = \lambda t$$

$$\mathrm{var}(N(t)) = \lambda t$$

Life distributions

Only two life distributions are introduced:

- The exponential distribution
- The Weibull distribution (with two parameters)

Several more life distributions are covered in the book.

Exponential distribution – 1

Consider an item that is put into operation at time $t = 0$. Assume that the time to failure T of the item has probability density function (pdf)

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t > 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is called the exponential distribution with parameter λ, and we sometimes write $T \sim \exp(\lambda)$.

The survivor function of the item is

$$R(t) = \Pr(T > t) = \int_t^\infty f(u)\, du = e^{-\lambda t} \quad \text{for } t > 0$$

The mean and the variance of T are

$$\text{MTTF} = \int_0^\infty R(t)\, dt = \int_0^\infty e^{-\lambda t}\, dt = \frac{1}{\lambda}$$

$$\text{var}(T) = 1/\lambda^2$$

Exponential distribution – 2

The failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The failure rate function is hence constant and independent of time.
Consider the conditional survivor function

$$R(x \mid t) = \Pr(T > t + x \mid T > t) = \frac{\Pr(T > t + x)}{\Pr(T > t)}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr(T > x) = R(x)$$

A new item, and a used item (that is still functioning), will therefore have the same probability of surviving a time interval of length t.
A used item is therefore stochastically as-good-as-new.

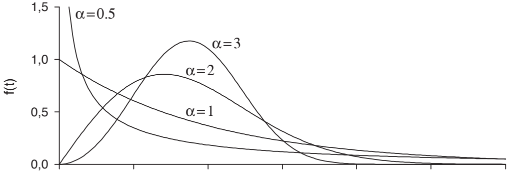
Weibull distribution – 1

The time to failure T of an item is said to be Weibull distributed with parameters α and λ [$T \sim \text{Weibull}(\alpha, \lambda)$] if the distribution function is given by

$$F(t) = \Pr(T \le t) = \begin{cases} 1 - e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding probability density function (pdf) is

$$f(t) = \frac{d}{dt}F(t) = \begin{cases} \alpha\lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$



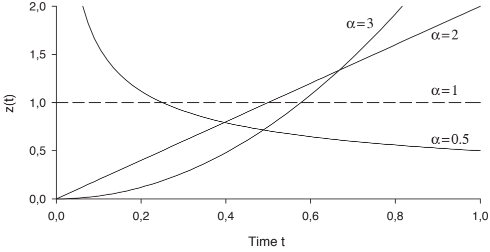
Weibull distribution (2)

The survivor function is

$$R(t) = \Pr(T > 0) = e^{-(\lambda t)^{\alpha}} \quad \text{for} \quad t > 0$$

and the failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \alpha \lambda^{\alpha} t^{\alpha - 1} \quad \text{for} \quad t > 0$$



Weibull Distribution (3)

The mean time to failure is

$$\text{MTTF} = \int_0^\infty R(t)\, dt = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha} + 1\right)$$

The median life t_{m} is

$$R(t_{\text{m}}) = 0.50 \quad \Rightarrow \quad t_{\text{m}} = \frac{1}{\lambda}\,(\ln 2)^{1/\alpha}$$

The variance of T is

$$\text{var}(T) = \frac{1}{\lambda^2}\left[\Gamma\left(\frac{2}{\alpha} + 1\right) - \Gamma^2\left(\frac{1}{\alpha} + 1\right)\right]$$

Note that $\text{MTTF}/\sqrt{\text{var}(T)}$ is independent of λ.